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Strategic changes in problem solving

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One way to study how individuals reason to solve problems is to see how they develop strategies to solve a series of related problems. This paper accordingly presents a theory explaining how they do so: When individuals solve a series of problems, their initial moves are constrained solely by perceptual and cognitive characteristics of the problems. They deduce the consequences of tactical moves, whether or not these moves are successful in advancing them towards a solution. As they master these tactics, however, a strategic shift occurs. The deduced knowledge comes to constrain the generation of moves, through the discovery of global constraints. Three experiments investigating a series of "matchstick" problems corroborated the theory.

Keywords: Deduction; Mental models; Problem solving; Reasoning; Strategies.

We reason to solve problems: For instance, we make deductions to tackle Sudoku problems (Lee, Goodwin, & Johnson-Laird, 2008). But, how does our reasoning improve our ability to solve problems? One way to answer the question is to track the mental steps that individuals take to solve a specific problem (e.g., Anzai & Simon, 1979). Another way, however, is to observe how individuals develop strategies as they tackle a series of related problems ("series problems"). The best-known series problems are those in which each problem calls for a quantity of water to be measured using only three jugs of fixed capacities (e.g., Luchins & Luchins, 1950). When individuals tackle these problems, they are likely to use a single deterministic procedure learned in earlier problems, even when a simpler solution exists (the so-called "set effect"). Lovett and Anderson (1996) also studied problems in which participants have to create a stick of a specified length, using any number of three other sorts of stick: One sort is longer than the target and two other sorts are shorter. When individuals tackle these problems, they begin with a perceptual strategy: If the longest sort of stick is similar in length to the target, then they start with a stick of this sort; otherwise they start with a shorter stick. However, the strategy fails for some problems, and over the course of several problems the participants learn the success rates of the various strategies. Subsequently, this knowledge guides their choice of a strategy (Lovett & Schunn, 1999).

We investigated a more sophisticated sort of series problem, i.e., "matchstick" problems. They call for seven distinct tactics; they cannot be solved using a single deterministic strategy; and they often have more than one solution (cf.

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Lovett & Anderson, 1996). A typical problem is shown in Figure 1a. It begins with an array of squares made from separate pieces, e.g., matchsticks. The task is to remove a specified number of pieces in order to leave a target number of squares in the array, but with two constraints: Each of the squares in the solution must consist of four pieces, and there must be no loose ends, i.e., pieces that are not part of a square. Three possible solutions are shown on the right of Figure 1a. Figure 1b shows the set of all seven possible tactics in matchstick problems: The tactics call for the removal of 1 to 4 pieces, respectively, and have the consequences of eliminating 0, 1, or 2 squares.

When individuals tackle series problems, they need to develop a *strategy*, that is, a sequence of tactical moves—tactics, for short—that yield a solution. Granted that problem solving is a computable process, only three sorts of strategy are possible (Johnson-Laird, 2006, Chap. 24):

(1) A *neo-Darwinian* strategy mimics evolution. It generates an arbitrary sequence of tactics, uses knowledge to deduce their consequences, and, depending on their viability, returns to the generative stage for further work, and so on, recursively.
(2) A *multistage* strategy is similar except that it uses some constraints based on knowledge to constrain the generation of sequences of tactics. For matchstick problems, such a constraint would be not to use a tactic that removes more pieces than the problem allows.
(3) A *neo-Lamarckian* strategy uses *all* the required constraints to generate a sequence of tactics, and so the sequence needs no revision and is a solution to the current problem. Thus, there is no need for recursion. When the constraints allow more than one possible tactic at a given step, an arbitrary choice is made among them.

Naïve individuals are unlikely to use a neo-Darwinian strategy to tackle matchstick problems. We implemented the strategy in a computer program, using an input of a problem with the two constraints: Each square must be made from four pieces, and there must be no loose ends.

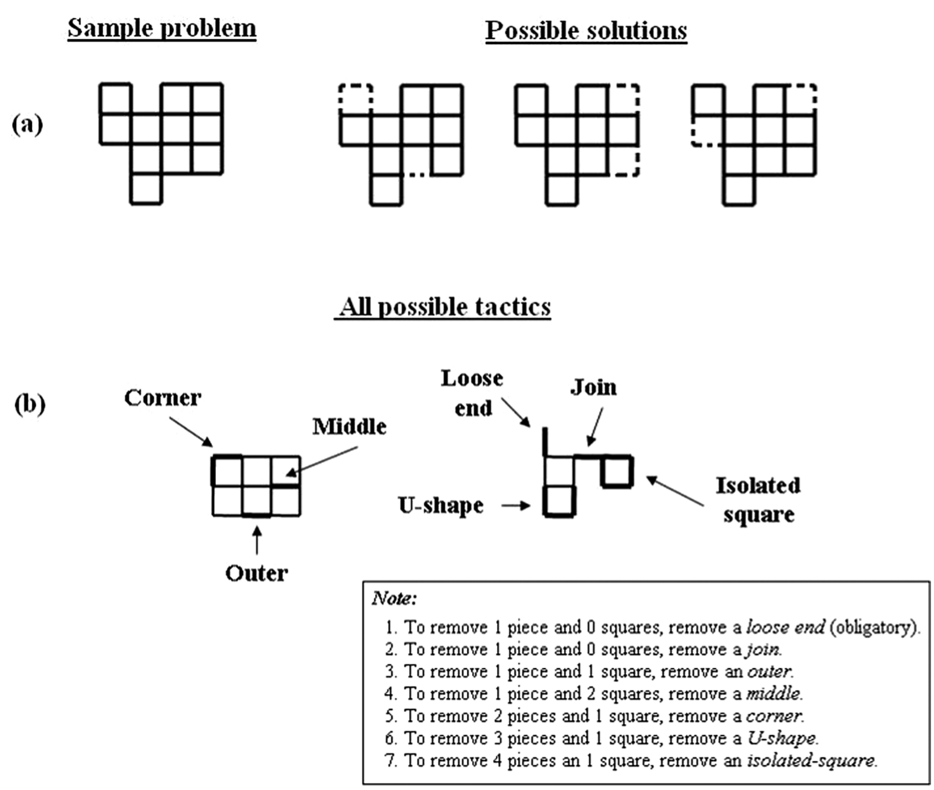

Figure 1. (a) A matchstick problem and its three solutions: "There are 11 squares in the shape. Please remove 4 pieces so that only 9 squares remain in the shape. There should not be any loose ends". (b) Seven possible tactics in matchstick problems: each tactic consists in the removal of one or more pieces and results in the elimination of 0, 1, or 2 squares.

It solved in tractable time only those problems that called for a small number of pieces to be removed. But, the strategy is intractable: As problems call for an increasing number of pieces to be removed, the search for a solution takes an exponentially greater amount of time. Given the solution, however, the time to check its correctness is only some polynomial of the number of pieces (Garey & Johnson, 1979).

Individuals are likely to use a more tractable strategy, which is constrained by perceptual processes. Two perceptual factors pertinent to matchstick problems are salience and symmetry. If certain pieces are more salient than others in the initial configuration of a problem (Feldman & Singh, 2005), then individuals should be biased to remove them. A pilot study showed that any piece in the perimeter of a problem is salient if it touches at least one piece at right angles to it in the perimeter. It will be even more salient if both adjacent pieces are at right angles to it, e.g., the bottom of the U-shape in Figure 1b. Likewise, if the initial configuration of squares is symmetric, then individuals should be biased to remove symmetrical pieces, because symmetrical pieces are salient (Wagemans, 1997), and participants can apply the same tactical moves twice.

Individuals discover new tactics by trying out different moves on a problem's configuration. As they try out a tactic, they deduce its consequences on the number of pieces it removes and the number of squares it eliminates. They learn these consequences whether or not the tactic helped them to solve the problem. Any small-scale problem allows only a limited set of tactics, and so individuals should gradually narrow down the sequences of tactics that are left to explore. Sooner or later, they should hit upon a sequence of tactics that leads to a solution. Thus, a cognitive change occurs: There is a *strategic shift* in which the deduced consequences of tactics, even unsuccessful ones, move from the evaluative stage of the process to the generative stage. In this way, they constrain the generation of tactics for problems later in the series. Individuals should accordingly develop a multistage strategy, and may even converge on a neo-Lamarckian strategy. Because of the variation in their exploration of tactics, and in the tactics each problem calls for, different individuals should develop different strategies. The shift of tactical knowledge should occur in any case, but a limited knowledge of tactics should yield limited performance in coping with future problems.

Knowledge of tactics provides individuals with local constraints on the generation of moves in particular circumstances. They may, however, discover global constraints that apply throughout the process of solving a problem. If they can thereby determine which tactic makes best progress towards solution, then they can create an optimal sequence of moves, i.e., they can employ a neo-Lamarckian strategy. They can even use a neo-Darwinian strategy in which the evaluative stage accepts a tactic only if it makes the best possible progress. The strategy will be wasteful because it generates many putative moves that evaluation rejects, but it will nonetheless solve the problem ultimately. In matchstick problems, a major global constraint is the ratio of the number of pieces-to-be-removed to the number of squares-to-be-removed (henceforth, the *pieces-to-squares ratio*). For example, if this ratio is much greater than one, then an appropriate tactic is to remove an isolated square or a U-shape (Figure 1b), whereas if the ratio is less than one, then an appropriate tactic is to remove a piece without eliminating a square. The present theory postulates that there should be a strategic shift of global constraints too.

EXPERIMENT 1

Our empirical research began with a study of local perceptual constraints. It tested the predictions that problems in which the removal of salient pieces leads to a solution should be easier than problems in which it does not, and that problems with symmetrical configurations should be easier than problems with asymmetric configurations.

Method, participants, and procedure

Twenty undergraduates at Princeton University carried out the eight problems shown in Figure 2 (Expt. 1). Each problem had a solution that was either salient or not, and an initial configuration that was either symmetric or not. One set of four problems (on the left of the figure) called for the same number of pieces and squares to be removed but had different initial configurations, and another set of four problems (on the right of the figure) had the same initial configurations but

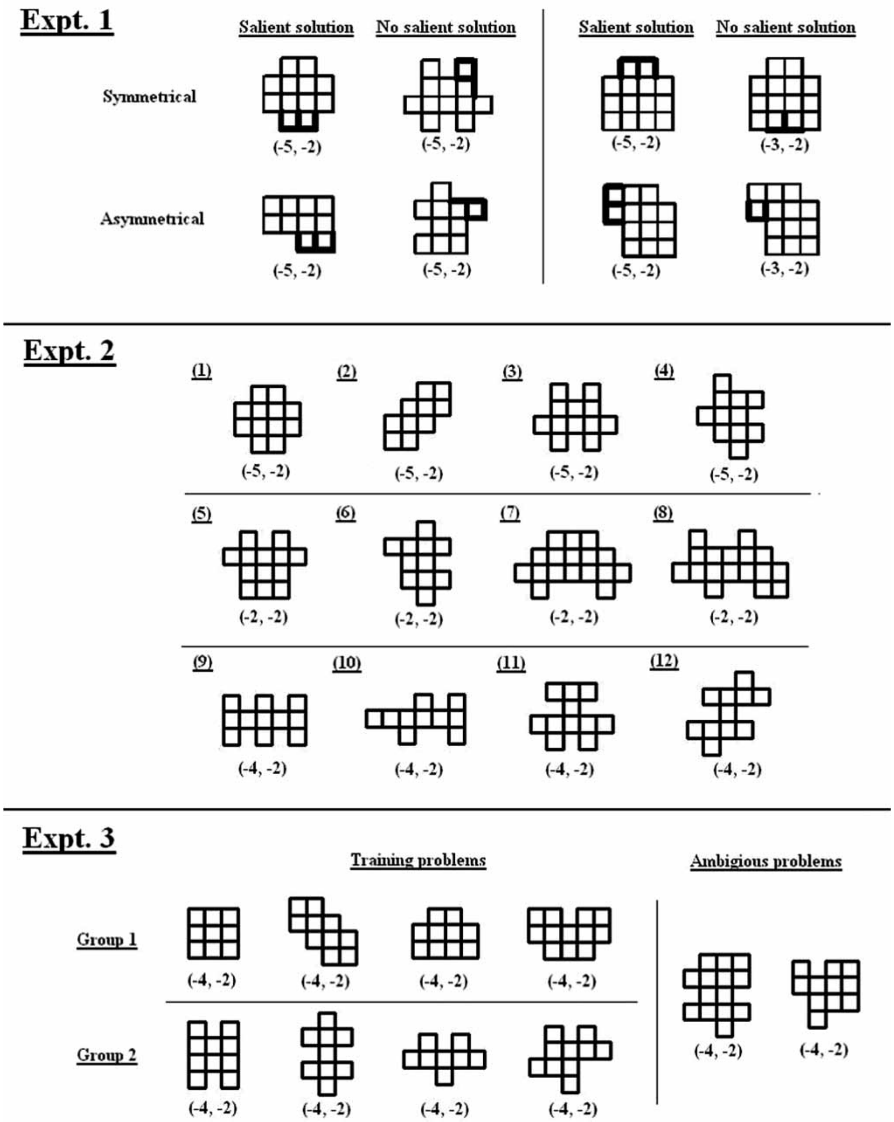

Figure 2. The problems in Experiments 1, 2, and 3. The two numbers in brackets are the required number of pieces to be removed and the required number of squares to be eliminated. Bold lines show the most frequent solutions in Experiment 1. In Experiment 3, participants first tackled the four problems in Group 1 (solution: remove two corners) or the four problems in Group 2 (solution: remove a U-shape and an outer), next the ambiguous problems that can be solved in either way, and then the final problem, which is a training problem for the other group.

different numbers of pieces and squares to be removed. The order of the two sets, and the order of the problems in each set, was random for each participant. On each trial, the participants first constructed the shape in a given diagram using matchsticks, and then they tried to solve the stated problem after it was revealed. They were told that each square in the solution must consist of four pieces and that there must be no loose ends. They had to say ‘‘done’’ as soon as they solved a problem.

Results and discussion

All the participants solved each of the eight problems. The two sets did not differ reliably in their latencies, and we therefore collapsed their latencies for analysis. The mean latencies and standard errors were as follows:

- symmetrical with salient solution: 16.7 s (2.3)
- symmetrical with no salient solution: 89.3 s (10.7)

- asymmetrical with salient solution: 26.0 s (18.7)
- asymmetrical with no salient solution: 119.5 s (17.5)

Problems with salient solutions were solved more than a minute faster than problems that lacked them (Wilcoxon test, $z = 3.92$, $p < .001$). Likewise, the participants solved symmetrical problems faster than asymmetrical problems (Wilcoxon test, $z = 2.09$, $p < .02$). The two variables did not interact reliably (Wilcoxon test, $z = 1.38$, $p = .17$, *ns*). Moreover, participants were more likely to create symmetrical solutions for the two problems that had both symmetrical and asymmetrical solutions. One problem had four possible solutions of which two were symmetrical, and 14 participants created symmetrical solutions (Binomial test, $p = .058$). Another problem had 20 possible solutions of which only one was symmetrical, and seven out of 20 participants created the symmetrical solution, which had an a priori probability of .053 (Binomial test, $p < .0005$). Thus, individuals tend to be constrained by salience and symmetry.

EXPERIMENT 2

This experiment tested whether individuals develop different strategies to solve matchstick problems, and whether they make the strategic shift. Almost any theory is likely to predict that over the experiment there should be a decline in the number of erroneous moves that individuals have to undo prior to solution, and a concomitant speeding up in solution times. A subtler and crucial prediction from the strategic shift is that the *proportion* of moves in solving a problem, up to the undoing of the last erroneous move, should diminish over the course of the experiment. Such a result would show that not only are participants acquiring knowledge of tactics, but also that this knowledge is undergoing a shift in which it comes to control the strategic generation of moves.

Method and procedure

Fourteen Princeton undergraduates carried out 12 problems in a different random order, and had to think aloud while they tackled them. The problems, which are presented in Figure 2 (Expt. 2), varied in terms of symmetry, number of pieces to be removed, and the required tactics, but all called for the removal of two squares. The instructions and procedure were the same as in Experiment 1, except that the participants had to think aloud, and the experimenter recorded what they said and what pieces they removed and replaced. If they fell silent for more than 3 s, the experimenter reminded them to think aloud. The protocols were subsequently transcribed verbatim for analysis. Since the think-aloud procedure is controversial (cf. Ericsson & Simon, 1984; Schooler, Ohlsson, & Brooks, 1993), we also carried out a replication of the experiment ($N = 14$) in which the participants did not have to think aloud.

Results and discussion

The participants in the experiment and its replication solved 93% and 98% of the problems respectively (Mann-Whitney $U = 54.5$, $z = 2.30$, $p < .05$, two-tailed), and so it was slightly harder to solve the problems and think aloud at the same time. But, no reliable difference occurred between the mean latencies to solve the problems between the two experiments (mean latencies 57.5 s and 51.5 s, Mann-Whitney $U = 77.0$, $z = .97$, *ns*). The rank-order difficulty of the problems in the two experiments was reliably correlated (Kendall's tau $= .49$, $p < .05$), and so the think-aloud requirement also had no reliable effect on the pattern of latencies over the course of the experiment. Table 1 presents the main results over the 12 trials in the experiment. The participants solved the problems increasingly faster over the 12 trials in the experiment and its replication (Page's $L = 7882.5$, $z = 4.86$, $p < .001$). As Table 1 also shows, the number of erroneous moves declined over the 12 trials of the experiment (Page's $L = 6727.5$, $z = 2.30$, $p < .025$). The number of erroneous moves correlated reliably with latencies for all but two of the problems (Pearson's r ranged from .65 to .95, with $p < .05$ to $p < .001$). And, as Table 1 shows, the proportion of moves prior to their last error on a problem shrank reliably over the course of the experiment (Page's $L = 5696.0$, $z = 2.60$, $p < .005$), although this general decline, as well as the decline in the number of erroneous moves, were not necessarily linear. The result corroborates the strategic shift.

The think-aloud protocols showed that twelve participants developed a two-stage strategy consistent with the hypothesised generative and

TABLE 1

The mean latencies (standard errors in parentheses), number of false steps (standard errors varied from 0.5 to 2.2), and proportion of moves prior to the last erroneous move that had to be undone (standard errors varied from 0.02 to 0.04), over the 12 trials in Experiment 2

Trial	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
Latency (s)	152.0 (41.8)	67.9 (16.9)	64.5 (18.8)	83.3 (24.7)	48.0 (10.0)	35.4 (12.8)	35.0 (8.8)	48.2 (15.8)	29.3 (7.8)	64.0 (26.5)	21.3 (4.4)	36.5 (12.3)
Number of false steps	5.8	2.9	3.9	5.2	4.4	1.2	2.0	4.0	1.3	3.4	0.9	1.9
Proportion of moves prior to last error	0.25	0.44	0.14	0.33	0.40	0.10	0.46	0.45	0.06	0.14	0.07	0.11

evaluative stages, and the length of each stage changed over the course of the experiment. Figure 3 presents an example of this two-stage strategy. Its details varied from one participant to another, but, during the first stage of tackling a problem, they explored various tactics and had to undo those that were erroneous. They were evidently generating tactics and then deducing their consequences. For example, they removed a middle piece, and then stated that its removal eliminated two squares without leaving any loose ends. They also deduced the effects on the pieces-to-squares ratio. Although the participants soon grasped the relevance of this ratio, they rarely relied on it in a complete way. They were evidently creating moves guided by only limited constraints. In a postexperimental questionnaire that examined knowledge of the seven possible tactics, the participants tended to identify only those tactics that they had used in the experiment, and no participant identified all seven tactics in Figure 1b.

During the second stage of tackling a problem, they relied to a greater extent on their tactical knowledge in the creation of moves, which led them directly to the solution. They had made the strategic shift. Towards the end of the experiment, the participants readily applied a tactic that they had acquired. They could combine some pairs of tactics into a single move that solved the problem at a stroke. They referred to the pieces-to-squares ratio much more often, and it constrained their performance to a greater degree. They could apply their knowledge recursively, removing an incorrect piece, recalculating the ratio, and, as a result, selecting a further tactic and so on, until they solved the problem. Hence, towards the end of the experiment, their strategies had become multistage procedures in which they used their knowledge to constrain their generation of moves. All but two of the participants who used the two-stage strategy made explicit reference to knowledge deduced from previous trials at least once. For example, one participant said, "OK . . . remove two matches and get rid of two squares again . . . um . . . could do the similar . . . in the same way I guess" (Participant 13, Trial 11). The optimal use of the pieces-to-square ratio depends on knowledge of the complete set of possible tactics, and no participant ever mastered all of them. Yet, some participants were able to solve problems towards the end of the experiment without making any erroneous moves.

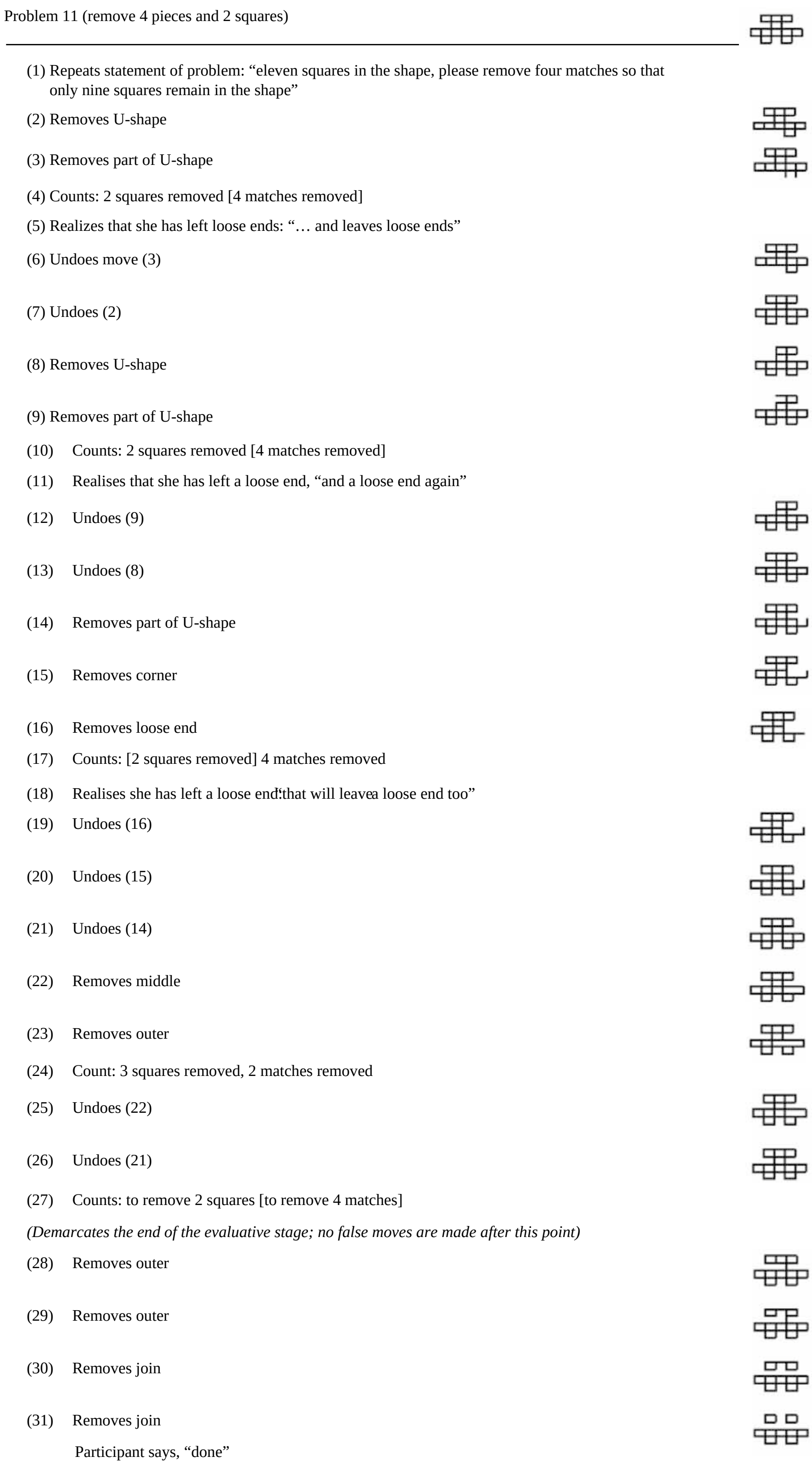

Figure 3. An abbreviated transcription of Participant 6’s think-aloud protocol and his corresponding sequence of tactical moves as he solved Problem 11 in Experiment 2. The transcription illustrates the main strategy that the participants used in the experiment.

The two participants who did not use the two-stage strategy developed their own idiosyncratic strategies. These strategies were inefficient, and one of these participants acquired no tactical knowledge whatsoever as shown in responses to the postexperimental questionnaire. The other participant did little better. Neither of these participants showed any signs of a strategic shift. They were able to solve most problems after many errors, and were not motivated to search for a more efficient strategy (cf. Lovett & Schunn, 1999).

EXPERIMENT 3

This experiment tested the prediction that a set effect should occur from the set of tactics that individuals acquire to the way in which they tackle subsequent problems.

Method and procedure

Twenty participants were randomly assigned to one of two groups. Group 1 first tackled four problems that could be solved only by removing two corners; Group 2 first tackled four problems that could be solved only by removing a U-shape and an outer. Both groups then tackled two ambiguous problems in a fixed order, which could be solved using either set of tactics, and a final problem chosen randomly from the first four problems given to the other group. Hence, this problem could not be solved using the tactics that the participants had previously acquired during the experiment. Figure 2 (Expt. 3) shows the problems, and the procedure and instructions were the same as those in Experiment 1.

Results and discussion

The participants solved 99% of the training problems, 90% of the ambiguous problems, and 85% of the final problems. They took progressively less time to solve the problems over the seven trials, with a mean of 57.1 s on the first trial and 43.9 s on the seventh and final trial (Page's $L = 1618.0$, $z = 4.23$, $p < .001$). But, they took significantly longer to solve the final problem, for which they had not acquired the appropriate tactics, than to solve the last training problem (mean 14.0 s vs. 43.9 s, Wilcoxon test, $z = 2.61$, $p < .01$). The choice of pieces that the participants removed for the ambiguous problems showed that both groups relied on the tactics that they had acquired in training: overall, 92% of solutions were based on these tactics; 14 out of the 20 participants used these tactics on both ambiguous problems, and the remaining participants were ties (Binomial test, $p < .5^{14}$).

Previous research (e.g., Luchins & Luchins, 1950) has shown that a set effect occurs when individuals acquire a single deterministic formula for solving series problems. The present results establish a different phenomenon. They show that individuals are susceptible to set effects that concern only knowledge of certain tactics, which are used in strategies that are neither formulaic nor deterministic. The participants accordingly succeeded in solving the ambiguous problems, but the final unambiguous problem impeded them because the tactics that they had acquired could not solve it, and so they were forced to discover new tactics.

GENERAL DISCUSSION

Although vast literatures exist for both reasoning and problem solving, their potential relations remain unclear. Some psychologists argue that reasoning and problem solving are unrelated (e.g., Robertson, 2001), and others consider reasoning to be an instance of problem solving (e.g., Newell & Simon, 1972). Our theory, however, establishes an important role for deduction in solving series problems, that is, problems of which many different instances exist. They include water-jug problems, Sudoku puzzles, diagnosis, troubleshooting, and many problems in daily life, from figuring out how to use an iPhone application to finding out how to contact an old friend. When individuals first tackle such a problem, they may be constrained only by the starting point, the goal, and the perceptual or cognitive characteristics of the problem. In matchstick problems, naïve individuals are strongly constrained not only by the salience of pieces but also by the symmetry of the configuration (Experiment 1).

As individuals gain experience in solving instances of a series problem, they deduce the consequences of various possible tactical moves, and they gradually develop a strategy to cope with the problems. The strategy is constrained by local characteristics, because deduced knowledge of tactical moves shifts to constrain the creation of moves. With further experience, they may acquire global constraints on performance.

With matchstick problems, they deduce a tactic's consequences for the pieces-to-squares ratio, which serves as a global constraint in creating new moves. The consequences are that they become more efficient in solving the series problems, and a telltale sign of the strategic shift is a reduction in the proportion of moves up to the final error in solving a problem (Experiment 2). Errors may even ultimately disappear if they can shift from a multistage strategy to a neo-Lamarckian one.

The strategic shift concerns only the tactical moves for which individuals have deduced the consequences. A corollary is a variant on the standard set effect in which individuals acquire a single deterministic routine to solve a series problem, such as the classical water-jug problems (Luchins & Luchins, 1950). They then overlook a simpler alternative routine for problems that allow it, and they may be stumped by problems that their routine cannot solve. Matchstick problems are different, because they are not susceptible to a simple deterministic routine. Individuals acquire knowledge of individual tactics, and they may acquire a strategy in which the pieces-to-squares ratio acts as a global constraint. The set effect in this case concerns tactical knowledge, so that they are biased to use certain tactics rather than others, and they are impeded when they encounter a problem that calls for new tactics (Experiment 3). This phenomenon generalises the standard set effect to problems that have no single deterministic strategy.

In principle, individuals could acquire a comprehensive knowledge of tactics, and use it to select sequences of tactics that would solve any given instance of a series problem. However, such a neo-Lamarckian strategy is unlikely to be feasible for every sort of series problem. The computational demands of some problems, such as writing a sonnet or a fugue, may overwhelm working memory. As a result, individuals have no option but to try out various tactical moves, and often to reject them. Theories of problem solving (e.g., Anderson & Lebiere, 1998; Newell, 1990) postulate that individuals who do not have specialised knowledge solve problems by relying on general heuristics, such as means–ends analysis. Our account is compatible with many sorts of computational architecture, such as SOAR and ACT-R, and it explains one role for deduction in problem solving. Individuals deduce the consequences of tactical moves, and thereby acquire knowledge. This knowledge, as part of a cognitive change, shifts to constrain the choice of tactical moves. Effective problem solving rests on the occurrence of this strategic shift.

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