Cognition 193 (2019) 103950

Contents lists available at ScienceDirect

Cognition

journal homepage: www.elsevier.com/locate/cognit

Original Articles

Possibilities as the foundation of reasoning

P.N. Johnson-Laird[a,b,*], Marco Ragni[c]

[a] Department of Psychology, Princeton University, Princeton, NJ 08540, USA
[b] Department of Psychology, New York University, 6 Washington Place, New York, NY 10003, USA
[c] Cognitive Computation Lab, Institut für Informatik, Technische Fakultät, Albert-Ludwigs-Universität Freiburg, Georges-Köhler-Allee, Geb. 052, 79110 Freiburg, Germany

ARTICLE INFO

Keywords:
Deontics
Mental models
Modal logics
Possibility
Probability
Reasoning

ABSTRACT

Reasoning about possibilities is fundamental in daily life. Yet, it has been little studied in psychology. We present a psychological theory in which it is the foundation of human reasoning. The theory explains how possibilities have distinct interpretations (deontic, epistemic, and alethic), how people represent them in models, and how these models yield inferences. Key principles are that the semantics of possibilities are the same finitary alternatives underlying probabilities, that speech acts can create obligations inexpressible as probabilities, that compound assertions – conditionals and disjunctions – refer to conjunctions of possibilities holding in default of knowledge to the contrary, and that mental models condense multiple consistent possibilities into one. The theory is incompatible with all normal modal logics and with probabilistic logic. Yet, experiments have corroborated its predictions. The article discusses its precursors, rivals, and potentials.

It is more than possible; it is probable.
–Sherlock Holmes

1. Introduction

Possibility can be close to probability, as Sherlock Holmes's remark bears out (see Conan Doyle, 1981, p. 339). Both introduce uncertainty into discourse, and so inferences about possibilities – *modal* reasoning – are ubiquitous in daily life. Here are three everyday examples, which we invite you to consider:

1. Either Trump will be re-elected or he won't be.
   So, it's possible that he won't be.
2. The probability of snow today is 90%.
   So, it's very possible that it will snow today.
3. Pat may be married and Viv may not be.
   So, maybe Pat is married and Viv isn't.

Modal logics deal with possibilities, and so you might wonder which of their main versions underlie these three inferences. The answer as we show is: none. The inferences above do not follow in any normal modal logic. Yet, naive reasoners – those who know nothing of logic – make them. You might think that these individuals are either blundering or else introducing additional, perhaps pragmatic, factors that justify the inferences. This article takes a more radical stance. It presents a theory of modal reasoning in which the inferences are *valid*, that is: if their premises are true then so too are their conclusions (Jeffrey, 1981, p. 1), but with one caveat that we explain below.

The theory distinguishes three principal interpretations of *possible* (for examples, see Table 1 below):

4. *Alethic* interpretations concern possible or necessary consequences.
   Deontic interpretations concern permissible or obligatory actions (or inactions).
   Epistemic interpretations concern degrees of belief in propositions.

These modalities have been studied in modal logic (e.g., Kripke, 1963), linguistics (e.g., Palmer, 2001), psycholinguistics (e.g., Miller & Johnson-Laird, 1976, Sec. 7.3.1), linguistic philosophy (e.g., White, 1975), formal semantics (e.g., Kratzer, 2012; Portner, 2009), and computational linguistics (e.g., Isard, 1975; Moens & Steedman, 1988). The new theory takes into account these studies. It is based neither on normal modal logics nor on their semantics of 'possible worlds,' but on a finitary semantics for *possibility* and its cognates. Reasoning, in turn, depends on models of possibilities. So, we refer to the new theory as the 'model' theory. It has a long-standing account of what reasoners are trying to compute in drawing their own conclusions (Johnson-Laird, 1983, p. 40; Johnson-Laird & Byrne, 1991, Ch. 2). They aim for conclusions that are new, parsimonious, and encapsulations of the semantic information in the premises. If no such conclusion exists, they respond

* Corresponding author at: Department of Psychology, Princeton University, Princeton, NJ 08540, USA.
E-mail address: phil@princeton.edu (P.N. Johnson-Laird).

https://doi.org/10.1016/j.cognition.2019.04.019
Received 21 August 2018; Received in revised form 3 April 2019; Accepted 22 April 2019

that nothing follows. If they are given a conclusion to evaluate, they check only that the conclusion maintains the premises' semantic information. So, unlike other accounts of reasoning, the theory distinguishes between conclusions that individuals accept and the subset of them that they draw for themselves. It treats all inferences in daily life as depending on tacit assumptions, and so the caveat that we mentioned earlier is that conclusions hold in default of knowledge to the contrary. Given such knowledge, they can be withdrawn. Reasoning is therefore *defeasible* (or *nonmonotonic*). When individuals withdraw a conclusion, they amend its premises, and try to explain the provenance of the inconsistency (Johnson-Laird, Girotto, & Legrenzi, 2004; Khemlani & Johnson-Laird, 2012).

The present article cannot describe the vast literature on modality, but it provides synopses of modal logic and modal linguistics (in Appendices A and B). Readers who suffer existential dread faced with logical symbols should read only the current state of the science (Section 2). The article next analyses the different interpretations of modal assertions, the cues to these interpretations, and their common underlying finitary semantics (Section 3). It shows how models can represent each interpretation (Section 4), and it illustrates how they yield different sorts of modal inference (Section 5). And it ends with a discussion of rival accounts, open questions for the model theory, and future lines of research (Section 6).

So, what do modals mean? How are they represented in the mind? And what mechanism uses these representations in modal reasoning? Our goal is to answer these three questions, relying in part on precursors. And, as we proceed, we enumerate the model theory's predictions and their empirical corroborations. They include long-standing results, and new tests of novel predictions.

2. Modality: The state of the science

Psychologists have studied modality for over fifty years (e.g., Byrnes & Beilin, 1991; Inhelder & Piaget, 1958; Piéraut-Le Bonniec, 1980; Shtulman, 2009), but know much less about it than about other sorts of reasoning. One hypothesis (Rips, 1994, p. 322) is that individuals can make deontic inferences without being familiar with their contents, e.g.:

5. It is obligatory that P given Q.
   Therefore, it is permissible that P given Q.

Rips suggested that such inferences are based on formal rules of inference with the addition of those for modal terms or of modal schemas for them (see Cheng & Holyoak, 1985). However, as he remarked, the extension of his system to deontic reasoning would entail more than just adding a few rules (Rips, 1994, p. 336).

Just what is at stake is illustrated in the major pioneering study of modal reasoning due to Osherson (1976). He formulated a theory of reasoning based on a modal logic (system T, see Appendix A). He constructed a language that has three sorts of modal operator: *possible*, *factual*, and *necessary*, but each sentence in the language contains only one of them. He described formal rules of inference for this language, and tested whether they could explain the relative difficulty of different inferences. As Osherson (1975) intended, the rules are incomplete, i.e., certain inferences valid in all normal modal logics cannot be proved in his system, including inferences of the following sort, which are implausible in daily life:

6. It's not the case that if it's raining then we'll get wet.
   Therefore, it's raining and we won't get wet.

He therefore introduced the concept of *psychological completeness*, which holds provided that a theory captures those valid inferences that untrained individuals make in ideal circumstances. As both he and Rips (1994, p. 124) argued, this sort of completeness can be settled only by observation and experiment. We go no further into details, because as Osherson (1976, p. 232) remarks, his results were not altogether successful. Participants did not accept all the valid inferences in a robust way.

In fact, previous studies of modality lack three essentials for a psychological theory. First, they have no plausible account of the meanings of modal assertions. They rely at best on a 'possible worlds' semantics (see Appendix A). On this account, an assertion such as:

7. It is possible that Trump is re-elected

is true if there is a relevant possible world in which he is re-elected. In each such possible world, any assertion about the real world is either true or false, and so as Partee (1979) remarked possible worlds are far too big to fit inside anyone's head. Second, current theories have no account of how the mind represents modal assertions. The model theory seems a plausible beginning (Bell & Johnson-Laird, 1998; Bucciarelli & Johnson-Laird, 2005; Goldvarg & Johnson-Laird, 2000), but, without the foundation of a proper semantics, it is suspended in mid-air. Third, current theories have no algorithm for the mental processes of modal reasoning. A countable infinity of modal logics exists (see Appendix A), but none allows the inferences in our three opening examples (1 to 3). These inferences also cannot be made in theories based on logic (e.g., Rips, 1994) or on probability (e.g., Oaksford & Chater, 2007).

3. What modals mean

3.1. Three interpretations of modals

Our first task is to distinguish among the different interpretations of modals in daily life, which have repercussions for reasoning. There are at least three main interpretations, which occur in this example in the order alethic, deontic, and epistemic:

8. It follows of necessity that she is permitted to resign if it's possible she wants to.

In this section, we show that these three interpretations are distinct, and we establish linguistic cues to them (cf. Kratzer, 2012). These components may vary from one language to another. Sentences in English and other Indo-European languages distinguish between possible and necessary, but not between deontic and other interpretations. An indigenous Salish language of British Columbia, St'át'imcets, is the other way round: it makes no distinction between possibility and necessity, but does distinguish between deontic and epistemic interpretations (Rullmann, Matthewson, & Davis, 2008).

3.1.1. Alethic interpretations

Alethic interpretations are always dyadic, i.e., they concern two sets of possibilities and the relation between them. In daily life, the possibilities concern inferences, causal antecedents and effects, and semantic analyses (e.g., Quelhas, Rasga, & Johnson-Laird, 2017). The two sets are usually epistemic or deontic, e.g.:

9. All the artists are beekeepers. So, necessarily, some of the beekeepers are artists.

But, the two sets of possibilities can themselves be alethic. For instance, the following asserts an alethic relation between two alethic relations:

10. A implies B and so not-B implies not-A.

Here and henceforth capital letters denote simple assertions or compounds containing conjunctions (based on 'and'), disjunctions (based on 'or') or conditionals (based on 'if … then …'). Modal auxiliaries can also assert alethic relations, e.g.:

11. All of the artists are beekeepers. So, all of the beekeepers may be artists.

If the premises imply the necessity of a conclusion, then the inference yielding the conclusion is valid. If a conclusion, *C,* follows only as a possibility, then the inference yielding *possibly C* is valid. Hence, a strong cue to an alethic interpretation is the occurrence of a sentential adverbial, such as 'necessarily', expressing the modality of a conclusion (as in 9). A necessary relation is that *A* implies *C*, which also has a conditional description, *If A then C*. Such a conditional refers to three possibilities:

A	C
¬ A	C
¬ A	¬ C

where '¬' denotes the negation of a clause. The relation in these possibilities can be stated in other alethic ways:

12. A makes C necessary.

The assertion:

13. A makes C impossible

is equivalent to *A* makes *not-C* necessary. And, the relation:

14. A makes C possible

can refer to all four possibilities, but is usually taken to mean that without *A*, *C* does not occur, and so to refer to three possibilities:

A	C
A	¬ C
¬ A	¬ C

We take up later the obvious relation between the assertions (12) to (14) and different sorts of causal claim (see 4.2.1.) As in all normal modal logics (see Appendix A), alethic necessity and possibility are interdefinable, though the negations may make the equivalence hard to grasp: if a conclusion is necessary then it is not possibly not the case, and if it is possible then it is not necessarily not the case.

Are alethic assertions on a scale? John Maynard Keynes (1921) thought so, and sought a probabilistic degree of implication, which presaged probabilistic logic (e.g., Adams, 1998). When individuals estimate the conditional probability that one assertion, *C*, holds given that another, *A*, holds, they can in principle estimate the proportion of cases of *A* in which *C* holds (Johnson-Laird, Legrenzi, Girotto, Legrenzi, & Caverni, 1999; Khemlani, Lotstein, & Johnson-Laird, 2012, 2015). They deduce a conditional probability. Likewise, they can deduce the conditional probability of a conclusion given that the premises hold. So, as Keynes argued, a probabilistic inferential relation does exist, which can be numerical or non-numerical:

15. From it's raining &/or cold, it follows with a high probability (of 2/3) that it is raining.

In sum, the knowledge relevant to alethic interpretations in daily life concerns the relations between two sets of possibilities, one for antecedents and one for consequences. Counterexamples, of course, reduce the probability and refute the necessity of conclusions.

3.1.2. Deontic interpretations

Deontic interpretations concern what is permissible, what is obligatory, and their negations. Their crucial characteristic is that speakers can create them. Utterances can bring into existence permissions, obligations, or prohibitions. These 'performative utterances' do something rather than just assert something (Austin, 1975, see, e.g., Lecture V). For instance, a policeman gives you permission by saying:

16. It is possible for you to leave now.

Utterances can be speech acts, because deontic states are based on social conventions – moral codes, customs, laws, rules, regulations – that enable agents to create deontic states. But, deontic utterances can also be mere descriptions, e.g., 'it was permissible for you to leave'.

A cue to a deontic interpretation is that 'permissible' can replace 'possible,' and 'obligatory' can replace 'necessary,' without changing the meaning of assertions. They then tend to use infinitival-complements or modal auxiliaries:

17. It is permissible for you to leave: You can leave.
    It is obligatory for you to leave: You must leave.

Although deontic assertions can use *that*-complements, their usage is less felicitous, e.g.:

18. It is permissible that you leave: You may leave.

Perhaps the reason for the infelicity is that *that*-clauses refer to propositions, but permissions apply, not to propositions, but to agents. Google ngram viewer corroborates the difference. For *is permitted to*, it shows a curve in usage from the years 1800 to 2000 with peaks well above zero. For *is permitted that*, it shows a flat curve barely above zero. Indeed, the first sentence in (18) cannot be used to give permission in a speech act. The first assertions in both examples in (17) contain an infinitival complement, 'for you to leave', in which the verb has no tense, and so the clause does not refer to a proposition. A proposition can be true or false, and so you can say:

19. It is true that you are permitted to leave.

but it is not acceptable for you to say:

20. It is true for you to leave.

Lacking a tense, infinitivals are missing information that would turn them into propositions. They are *propositional functions*, which given certain additional information yield propositions. One way to turn them into propositions is to stipulate their deontic status (as in 16). They are then propositions because they can be true or false:

21. It is true that it is possible for you to leave.
    A corollary is that the possibility of a proposition implies the possibility of its propositional function:
22. It is possible that Queen will abdicate.
    So, it is possible for the Queen to abdicate.

But, the converse inference does not follow of necessity: it may be possible for her to abdicate, but impossible that she does (out of a sense of duty).

Deontic interpretations rest on knowledge. For example, a tennis umpire's call of 'let' after a serve obligates the player to serve again. Like alethic modals, deontic possibility and necessity are interdefinable using negation. But, unlike alethic modals, deontic interpretations are 'all or none', because one cannot give a graded permission, such as:

23. It is almost permissible for you to leave.

This assertion is not a speech act giving a graded permission, but rather a description of a situation in which the listener has almost fulfilled the conditions needed for permission to leave.

A final cue to deontic modals concerns counterexamples. A counterexample to an obligation violates it, but has no bearing on the truth of the obligation (Wertheimer, 1972). So, a failure to keep the tenth Commandment does not thereby refute it.

3.1.3. Epistemic interpretations

Epistemic interpretations reflect the use of knowledge to assess the credibility of propositions, and so they are similar, if not identical, to subjective probabilities. As we showed above, *that*-clauses refer to propositions, but infinitivals refer to a broader notion of a propositional function. The difference parallels a difference in the usage of modal auxiliaries in epistemic assertions:

24. It is possible that the tsunami will flood us: the tsunami may flood us.

and:

25. It is possible for the tsunami to flood us: the tsunami can flood us.

The inferential relation between *that*-clauses and infinitivals, which we established for deontic interpretations, holds for epistemic interpretations too. But, the critical inference is from the negation of the possibility of a propositional function to the negation of the corresponding proposition:

26. It is not possible for the tsunami to flood us.
    So, it is not possible that the tsunami will flood us.

This conclusion, in turn, implies:

27. The tsunami will not flood us.
    Assertions of necessity, such as:
28. It is necessary for Trump to be indicted.
    It is necessary that Trump is indicted.

do not imply that he will be indicted. They state a necessary condition for some other state of affairs, such as the good of the country, and so it therefore makes sense to assert:

29. It is necessary for Trump to be indicted but not possible that he will be.

What is striking about epistemic possibilities is their kinship with subjective probabilities (see Appendix B). Epistemic 'possible' has an interpretation exemplified in the following scale with a lower and an upper bound in which 'certain' occurs rather than 'necessary' as the upper bound:

30. Impossible – almost impossible – possible – very possible – almost certain– certain

It is akin to the scale of non-numerical probabilities (von Mises, 1928, p. 62):

31. Impossible – very unlikely – unlikely – as likely as not – likely – very likely – certain

Actual numerical estimates of the probabilities corresponding to the terms in the two preceding scales bear out their scalar interpretation (e.g., Juanich, Teigen, & Gourdon, 2013; Mosteller & Youtz, 1990). Lassiter (2017) has argued that a single ratio scale exists combining both epistemic possibilities and probabilities. In any case, it is simple to translate from one scale to the other, and to make inferences of the sort exemplified in our opening example (2):

32. The probability of snow today is 90%.
    Therefore, it is very possible that it will snow today.

In some modal logics, any string of modal operators can be reduced to the final one in the string (see Appendix A). Everyday modality, however, is different. In the case of epistemic claims, such as:

33. Perhaps, it's possible that Biden may win the nomination

the three modals – 'perhaps,' 'possible,' and 'may' – suggest tentativeness about the proposition.

They appear to move it towards the less probable end of the scale. These modal terms cannot be reduced to a single one.

3.1.4. A summary of interpretations

There are multiple cues to the three main interpretations of modal assertions. Table 1 summarizes four cues that distinguish each of them. A decisive cue to a deontic interpretation is the occurrence of speech act creating a deontic state, e.g.: 'It is permissible for you to leave'. For many assertions, more than one interpretation is feasible even in context. For example, a fire chief who tells the residents of a building after an emergency:

Table 1
Three sorts of interpretation of modal assertions (alethic, deontic, and epistemic), with examples of each, and four cues that distinguish them, where 'yes' indicates that the cue holds and 'no' indicates that it does not hold. The scale for alethic modals is from impossible to necessary, whereas the scale for epistemic modals is from impossible to certain.

Sorts of modal	Alethic	Deontic	Epistemic
Examples	It follows as a possibility that you left.	It's possible for you to leave.	It's possible that you left.
Cue 1: Speech acts can create possibilities	No	Yes	No
Cue 2: Interdefinability of possible and necessary	Yes	Yes	No
Cue 3: Must be dyadic	Yes	No	No
Cue 4: Scalar, and an end point refutable by a counterexample	Yes	No	Yes

34. You may go back into the building now

could be granting permission or describing an epistemic possibility, or both. Such indeterminacies should occur given that modals share an underlying meaning.

3.2. The underlying meaning of possibility

A crux is whether a modal term such as *possible* has different meanings as *bank* has in 'He put money in the bank' and 'He sat on the river's bank', or different interpretations of a single underlying meaning, as *run* has in 'He went for a run' and 'He hit a run' (Johnson-Laird, 1978; Kratzer, 1977). In the model theory, *modulation* by relevant knowledge yields different interpretations of a single underlying semantics (Johnson-Laird & Byrne, 2002). Its mechanism conjoins models of premises with fully explicit models representing knowledge (see Khemlani, Byrne, & Johnson-Laird, 2018; and Appendix C below). Indeterminate interpretations occur when different bodies of knowledge modulate the same assertion. That is why it can be difficult to decide which interpretations of a modal assertion are appropriate.

The model theory postulates that the ability to consider alternatives underlies modality. Human beings and some other species can

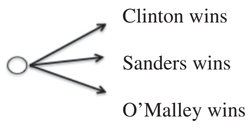

Fig. 1. The three alternative outcomes for the Iowa caucuses on February 1st 2016.

○ Fact: Clinton wins
♦ Counterfactual possibility: Sanders wins
♦ Counterfactual possibility: O'Malley wins
Inaccessible impossibility: Elizabeth Warren wins

Fig. 2. The status of the prior alternatives after the outcome of the Iowa Democratic caucuses on February 1st 2016. The circle denotes a fact, and the black triangles denote counterfactual possibilities.

categorize almost any situation into a small finite number of exhaustive and mutually exclusive alternatives, and the ability is evident in early childhood (Redshaw et al., 2019). The Iowa Democratic caucuses on February 1st 2016 could have occurred in many different ways depending on what each delegate said to others, how the delegates divided up at first, the results of further interchanges of views, and so on and on. An infinite number of possibilities could have occurred. Yet, there are only three alternative outcomes, and Fig. 1 shows them. Of course, you may envisage other remote outcomes – the caucus is never completed as a result of an act of God. But, for most purposes, your concern is with the three outcomes. This ability to categorize into finite alternatives underlies the semantics of possibilities and probabilities:

35. A possibility in daily life is a subset of finite mutually exclusive and exhaustive alternatives, where each alternative is a category stipulating what is common to an indefinite number of different realizations.

For example, one subset of the alternatives in Fig. 1 contains two of its members:

36. Possibly, Clinton or Sanders will win the caucus.

An outcome that is an alethic necessity refers to the complete set of alternatives:

37. Necessarily, Clinton, Sanders, or O'Malley, will win the caucus.

Given the actual results of the caucus, the status of the possibilities changes. Only one individual can win, so one of the three alternatives becomes a fact, and the other two become *counterfactual* possibilities, i.e., outcomes that were once possible but that did not occur (Johnson-Laird & Byrne, 2002). A corollary is a major principle of the model theory:

38. Each possibility holds in default of knowledge that rules it out, converting it into a counterfactual possibility.

There are also cases that are impossible, e.g., Elizabeth Warren could not have won the Iowa caucuses of 2016, because she was not a candidate. Fig. 2 shows the change to the possibilities given the outcome of the caucuses. Speakers can refer to the counterfactual cases using a counterfactual conditional (see Byrne, 2005; Byrne, Goodwin, Johnson-Laird, Khemlani, Quelhas, & Ragni, 2019), as in:

39. If Sanders had won in Iowa then he might have won the nomination.

Normal modal logics do not distinguish between counterfactual possibilities and real possibilities. Hence, as Karttunen (1972) pointed out, there are discrepancies between logical but infelicitous descriptions, such as:

40. Sanders lost, but he may not have.

and counterfactual and felicitous descriptions:

41. Sanders lost, but he might not have.

The translation of this last assertion into a normal modal logic is equivalent to asserting that Sanders lost but it is possible that he didn't lose, which is false in the logic. Yet, assertion (41) is not false, and so such examples diverge from normal modal logics.

The finite semantics underlies the different interpretations of modals, which result from modulation. Knowledge of relations between sets of possibilities underlies an alethic interpretation. It concerns the possibilities to which an antecedent refers and another set to which a consequence refers. We detail these relations in outlining the theory of modal reasoning (in Section 5.1). For deontic interpretations, modulation implies that alternatives refer to what is permissible, and that actions that violate an obligation do not refute it. It also implies, unlike an alethic interpretation, that necessity does not imply actuality, because people do not always carry out their obligations. Knowledge that establishes the credibility of a proposition yields an epistemic interpretation. For a true assertion, *it is possible that A*, *A* holds in a subset of the alternative outcomes in the relevant situation.

A major principle of the model theory is:

42. An assertion of any of the three sorts of possibility presupposes the possibility of its negation, where a presupposition has to be true for a sentence to assert a proposition, i.e., to be true or false, e.g., 'He stopped smoking' presupposes that he smoked.

So, an alethic assertion:

43. It follows as a possibility that you left

presupposes that it also follows that as a possibility you did not leave. A deontic assertion, such as:

44. It is possible for you to leave

presupposes that it is possible for you not to leave. And an epistemic assertion:

45. It is possible that you left

presupposes that it is possible that you did not leave.

3.3. Sentential connectives depend on modals

Sentential connectives such as *if* and *or* form compound assertions, and a fundamental assumption of the model theory (Khemlani et al., 2018) is:

46. Compound assertions, such as conditionals and disjunctions, refer to an exhaustive conjunction of possibilities that hold in default of knowledge to the contrary.

For instance, an inclusive disjunction:

47. A or B, or both.

refers to these exhaustive possibilities:

48. Possible (A and not B) and possible (not A and B) and possible (A and B).

They exhaust the possibilities to which the disjunction refers, and so they imply what is impossible given the truth of the disjunction:

49. Not possible (not A and not B).

When the facts eliminate all of a compound's default possibilities, it is false. A conjunction of the sort, *A and B*, is also a compound assertion, but it refers to only one case, and so it makes a factual claim rather than one about a possibility.

A tempting misinterpretation of the model theory is that its meanings for connectives are equivalent to truth tables (described in Appendix A). But, cases in a truth table are mutually exclusive, e.g., a case of *A and not-B* cannot be conjoined with a case of *A and B* without contradiction. In contrast, the corresponding possibilities are consistent with one another, as they are in (48). The model theory therefore implies:

Prediction 1. Naive individuals should interpret compound assertions, such as disjunctions and conditionals, as referring to conjunctions of default possibilities, even if these compounds make no explicit references to possibilities.

Given an inclusive disjunction, such as:

50. The flaw is in the software or it is in the cable, or both of them.

individuals in an experiment accepted each of the following epistemic possibilities as conclusions on separate trials (Hinterecker, Knauff, & Johnson-Laird, 2016):

51. It is possible that the flaw is in the software.
It is possible that the flaw is in the cable.
It is possible that the flaw is in the software and in the cable.

The computer implementation of the theory provided an accurate fit to these and other data from the experiment (Khemlani, Hinterecker, & Johnson-Laird, 2017). A similar inference occurs in our opening example (1):

52. Either Trump will be re-elected or he won't be.
So, it's possible that he won't be.

The premise yields a conjunction of two default possibilities:

53. Trump re-elected
¬Trump re-elected

where as usual '¬' denotes negation. As these models show, the conclusion follows at once. For most people, the preceding inferences are obvious, and some critics have even argued that their validity is obvious too. Yet, they are invalid in all normal modal logics. Suppose for the inferences from (50) that it is impossible that the flaw is in the software. The first of the three possibilities above in (51) would be false. Yet the disjunction would still be true in modal logics if the flaw is in the cable. And so the disjunction could be true, but the first conclusion in (51) false. An obvious way to guarantee its possibility in normal modal logics is to add the premise:

54. It is not impossible that the fault is in the software.

But, this premise is equivalent to the possibility in the required conclusion, and so the inference would be circular. A similar problem occurs with the inference in (52). If it is impossible that Trump won't be re-elected then the premise is true in normal modal logics, but the conclusion is false. Hence, such logics and mental models differ in a striking way. Similar phenomena occur with conditionals (e.g., Byrne et al., 2019; Byrne & Johnson-Laird, 2019; Goodwin & Johnson-Laird, 2018). But, studies have examined only compounds yielding epistemic possibilities, and so an open question is whether the predicted phenomena occur with deontic and alethic compounds.

4. How modals are represented in the mind

4.1. Mental models and fully explicit models

The model theory postulates that a mental model is a small finite representation that is iconic, i.e., insofar as possible it has the same structure as what it represents. So, each of the alternative outcomes of, say, the Democratic Iowa caucus in Fig. 1 can be represented in a model of a possibility. The theory postulates that a parser uses a grammar and a lexicon to compose the meanings of assertions. An intuitive system of reasoning, system 1, uses these meanings to construct a *mental model* of the situation. System 1 cannot use working memory to store the results of intermediate computations. It yields intuitive conclusions based on a single mental model at a time. The mental models of an inclusive disjunction, such as:

55. Albert is at the party or else Betty is, or both of them are

represent the individuals who may be at the party. Hence, the following diagram denotes three mental models, on separate lines, which represent the conjunction of default possibilities about who is at the party:

Albert	
	Betty
Albert	Betty

Because each possibility holds in default of knowledge to the contrary (see 46), only if knowledge overrules the presence of Albert and the presence of Betty is the disjunction false. Of course, real mental models represent the world, and our diagrams use words for convenience. But, as the example illustrates, a long-standing principle of the model theory (e.g., Johnson-Laird & Savary, 1999) is:

56. The principle of truth: mental models represent only those possibilities in which an assertion is true, and in each of these models they represent only those clauses in the assertion that are true in the relevant possibility.

System 2, a deliberative system of reasoning, has access to working memory, and so it can flesh out mental models into fully explicit models, using negations to do so, as in these models for disjunction (55):

Albert	¬ Betty
¬ Albert	Betty
Albert	Betty

System 2 can examine alternative models, find counterexamples to system 1′s conclusions, and draw its own conclusions. These conclusions are correct according to the theory, that is, they are intended to embody a psychologically complete theory of reasoning from modals and sentential connectives (see Section 2. for this concept of psychological completeness).

The idea of two systems for reasoning is due to the late Peter Wason, and it was implemented first in an algorithm for a theory of the selection of evidence to test hypotheses (Johnson-Laird & Wason, 1970; see also Ragni, Kola, & Johnson-Laird, 2018, which shows that this algorithm is more accurate than subsequent accounts of Wason's selection task). His students developed such 'dual process' theories of reasoning (e.g., Evans, 2008; Johnson-Laird, 1983, Ch. 6), but others have also formulated them too (e.g., Kahneman, 2011; Stanovich, 1999).

The principle of truth (56) implies that mental models represent what is true according to the premises. The result is parsimonious. However, it has an unexpected consequence: mental models yield compelling illusory inferences from certain premises (Johnson-Laird & Savary, 1999; Khemlani & Johnson-Laird, 2017). We illustrate them for modal reasoning later (in Section 5).

4.2. Mental models of possibilities

Models can represent possibilities of all three sorts – alethic, deontic, and epistemic. They can also represent different sorts of discourse – factual, counterfactual, hypothetical, and fictional, and on occasion the probabilities of propositions (Johnson-Laird, 1983, p. 60). The theory therefore assumes that symbols can attach to models to signify their status, just as negation is represented by a symbol that has an appropriate semantics (e.g., Johnson-Laird & Byrne, 1991, p. 68-9). For simplicity, when matters are clear, we forego such symbols.

4.2.1. Alethic possibilities

Alethic possibilities may concern premises and conclusions, concepts and their analyses, and antecedents and effects of causal relations, such as *causes*, *prevents*, and *allows*. As our studies corroborate, *prevents from happening* is equivalent to *causes not to happen*, and so we focus here only on *causes* and *allows*. The following two assertions:

57. Building the canal will cause the city to flood.
 Building the canal will allow the city to flood.

refer to temporally constrained sets of possibilities. The temporal constraint is that in daily life the antecedent of a causal relation cannot occur after its effect. Matters can be different in philosophy and quantum gravity. Causing refers to a *necessary* alethic relation between building the canal and flooding (3.1.1.):

canal	flooding
¬ canal	flooding
¬ canal	¬ flooding

And allowing refers either to all four possibilities between building the canal and flooding or to a *possible* relation:

canal	flooding
canal	¬ flooding
¬ canal	¬ flooding

So, causing an event makes it necessary, and allowing an event makes it possible. This account goes back to Miller and Johnson-Laird (1976, Section 6.3).

The theory implies that causation is deterministic rather than probabilistic, which Pearl (2009), a pioneer of Bayesian networks, defends, and which empirical studies corroborate (e.g., Frosch & Johnson-Laird, 2011). Probabilities enter only because evidence for a causal relation can be uncertain. The model theory yields:

Prediction 2. Naive individuals should list conjunctions of possibilities that discriminate among assertions of the sort: *A causes B, A prevents B,* and *A allows B*.

Experiments have corroborated this prediction with assertions, such as:

58. Running the new application will cause the computer to crash.The assertions occurred with each causal verb on separate trials, and the results corroborated the predictions (Goldvarg & Johnson-Laird, 2001). Likewise, when participants had to make mental simulations to infer whether one event caused or allowed another to occur, they distinguished the two, and their latencies depended on the number of moves called for in the simulation (Khemlani, Goodwin, & Johnson-Laird, 2015).

Many theories deny that causing and allowing differ in meaning. The claim goes back to Mill (1874, p. 236 et seq.), it infected philosophers and jurists – common law does distinguish the two – and then cognitive scientists. The latter have argued that causal conditions are rare, inconstant, and relevant to descriptions, whereas allowing conditions are common, constant, and irrelevant to descriptions. Yet, these claims are refutable, and causing and enabling do refer to different possibilities (Khemlani, Barbey, & Johnson-Laird, 2014). And, as holds for any claim about alethic necessity – in causation (Frosch & Johnson-Laird, 2011) or in deduction (Johnson-Laird & Hasson, 2003) – a single counterexample suffices to refute a necessary relation.

4.2.2. Deontic possibilities

Deontic possibilities can be represented in mental models. As our studies corroborated, the prohibition of an action is equivalent to the obligation not to carry it out, and so we focus here only on obligation and permission. Assertions, such as:

59. Tax-payers who support charities are obligated to claim a rebate on their taxes.
 Tax-payers who support charities are permitted to claim a rebate on the their taxes.

Describe a condition for a deontic state. The obligation in (59) has the following models of deontic possibilities, where *t* denotes tax-payers who support charities and *c* denotes those claiming a rebate on their taxes:

t	c
¬ t	c
¬ t	¬ c

The permission in (59) has models of either all four deontic possibilities or the following three:

t	c
t	¬ c
¬ t	¬ c

As a consequence, the model theory yields:

Prediction 3. Naive individuals should list conjunctions of deontic possibilities that differ for assertions of the sort: A obligates B, A prohibits B, and A permits B, where A denotes an agent's action (or its omission) and B denotes a deontic state.

Experiments corroborated this prediction using everyday sentences, such as those in (59). The participants tended to list the predicted possibilities (Bucciarelli & Johnson-Laird, 2005).

4.2.3. Epistemic possibilities

On the assumption of roughly equal possibilities, models of the alternatives in Fig. 1 imply:

60. It is likely that Clinton or Sanders will win.

Participants make such estimates, and labels on models can represent numerical probabilities (Johnson-Laird et al., 1999). In epistemic conditionals of the sort, *If A then B*, the *if*-clause is subordinate to the main *then*-clause, and their meaning can be paraphrased as:

61. It is possible that A, and in this case B

where the initial clause presupposes (see 42):

62. It is possible that not-A.

One consequence is that the mental models of a conditional do not represent the cases in which its *if*-clause is false, because they are possible whether the conditional is true or false. Hence, a conditional such as:

63. If Donald is in the office then Kellyanne is in the office.

has the following mental models:

Donald in the office	Kellyanne in the office
	…

where the ellipsis denotes the presupposed possibility that Donald is not in the office. These models can be fleshed out in system 2 into fully explicit models:

Donald in the office	Kellyanne in the office
¬ Donald in the office	¬ Kellyanne in the office
¬ Donald in the office	Kellyanne in the office

We state them in the order in which children tend to acquire them and adults tend to list them (e.g., Barrouillet, Grosset, & Lecas, 2000). The last two models above represent the conditional's presuppositions. Because the possibilities are exhaustive, there is one impossibility:

Donald in the office	¬ Kellyanne in the office

The empirical consequences of this account of conditionals have been examined in detail elsewhere (see Byrne et al., 2019; Goodwin & Johnson-Laird, 2018; Khemlani et al., 2018), and so we only sketch them here. The denial (i.e., negation) of a conditional such as (63) depends on whether its presuppositions are taken into account. Without them, its negation is:

64. Donald is in the office and Kellyanne is not in the office.

With them, it is:

65. If Donald is in the office then Kellyanne is not in the office.Experiments have corroborated the occurrence of these two sorts of denial (Khemlani, Orenes, & Johnson-Laird, 2014). Cases in which a conditional, *if A then B*, has a false presupposition (i.e., cases of *not-A*) do not provide evidence about whether the conditional is true or false, because they are possible in either case. Hence, the conditional (63) is true provided that it is possible that Donald and Kellyanne are in the office and that it is impossible that Donald is in the office and Kellyanne is not. The theory therefore reconciles two discrepant sets of findings. On the one hand, individuals judge that the *not-A* cases are irrelevant to the verification of *If A then B* (e.g., Johnson-Laird & Tagart, 1969). They do so, because these cases hold whether it is true or false. On the other hand, they list them as possible for a conditional (e.g., Barrouillet et al., 2000). They do so, because they are indeed possible. Their irrelevance to the verification of conditionals also yields the well-known Equation (with a capital 'E') according to which the probability of a conditional, *If A then B*, equals the conditional probability of *B* given *A* (see, e.g., Adams, 1998).

Epistemic possibilities yield a scale corresponding to degrees of belief. According to the model theory, it uses the same System 1 mechanism that yields non-numerical but ordinal probabilities, such as 'hardly likely', and 'highly probable', for unique events, e.g.:

66. The probability that Trump will run in the Presidential election of 2020.

The mystery in such estimates is where the numbers come from; an answer, its computer implementation, and corroboratory evidence, are described in Khemlani et al. (2012, 2015). In essence, the proportion of the models of pertinent evidence in which the relevant event occurs shifts the probability of the event up or down on a non-numerical scale. System 1 can translate its value into an informal description, such as 'highly probable' and system 2 can translate it into a numerical estimate, such as '80% likely'. Estimates of the probabilities of, say, *A*, *B*, and *A and B*, fix the joint probability distribution, i.e., the probabilities of each member in the partition: *A and B*, *A and not-B*, *not-A and B*, and *not-A and B*. But, the process underlying estimates should tend to be subadditive, and so, contrary to the probability calculus, the probabilities of the cases in the partition should sum to more than 100%. Experiments corroborated this prediction (Khemlani, Lotstein et al., 2015). And, when participants estimate the joint probability of a conditional with each member of its partition, then the sum of these estimates should be grossly subadditive, because a subadditive process makes four estimates. Indeed, their means in experiments summed to over 200% (Byrne & Johnson-Laird, 2019; Goodwin & Johnson-Laird, 2018).

When an assertion makes an explicit reference to an epistemic possibility of a proposition, as in:

67. It is possible that Donald is in the office.

it presupposes the possibility of the negation of the proposition (see 42). Hence, the mental models of (67) represent a conjunction of two default possibilities:

Donald in the office
…

where the ellipsis is an implicit model of the presupposition of the possibility that Donald is not in the office. Likewise, the mental models of a possible disjunction, such as:

68. It is possible that Jared or Ivanka is in the office, or both of them are has the following mental models of who is in the office:

Jared	
	Ivanka
Jared	Ivanka
…	

The ellipsis denotes the presupposition of the possibility that the disjunction does not hold, i.e., the possibility that neither Jared nor Ivanka is in the office. Table 2 illustrates the mental models and the fully

Table 2
The mental models and fully explicit models for representative categorical and compound assertions and for possibilities of compounds. A true assertion with a single model represents a fact; otherwise, each set of models for a true assertion represents an exhaustive conjunction of mutually exclusive possibilities that hold in default of knowledge to the contrary. The ellipsis, …, refers to the presupposition that the relevant proposition does not hold, and the symbol, ¬, refers to sentential negation.

Assertions	Mental models		Fully explicit models		
A and B	A	B	A	B	
A or else B but not both	A		A	¬ B	
		B	¬ A	B	
A or B	A		A	¬ B	
		B	¬ A	B	
	A	B	A	B	
If and only if A then B	A	B	A	B	
	…		¬ A	¬ B	Presupposition
If A then B	A	B	A	B	
	…		¬ A	¬ B	Presupposition
			¬ A	B	Presupposition
It is possible that A	A		A		
	…		¬ A		Presupposition
It is not possible that A	¬ A		¬ A		
It is possible that A or B	A		A	¬ B	
		B	¬ A	B	
	A	B	A	B	
	…		¬ A	¬ B	Presupposition
It is possible that if A then B	A	B	A	B	
	…		¬ A	¬ B	Presupposition
	…		¬ A	B	Presupposition
			A	¬ B	Presupposition

explicit models for various sorts of assertion, including those containing sentential connectives.

5. What mechanisms use models for modal reasoning

The model theory applies to all sorts of reasoning including induction, abduction, and deduction (Johnson-Laird, 2006; Khemlani & Johnson-Laird, 2013). It depends on simulating the world in models of possibilities, either static or kinematic (Johnson-Laird & Byrne, 1991; Khemlani, Mackiewicz, Bucciarelli, & Johnson-Laird, 2013). We have implemented the present theory of deduction in the mSentential program at http://mentalmodels.princeton.edu/models/. The program ensures that the model theory does not take too much for granted, and it determines the status of any inference from a potential infinity of them. But, we focus here, not on the program, but on the model theory itself.

The basic mechanism for reasoning calls for the pairwise conjunction of the sets of models for each of the premises, and the fundamental principle is that a conjunction of one model of a possibility with another proceeds unless an element in one of the models contradicts an element in the other. For readers who want to know how to construct models, Appendix C describes the process.

The mechanism for reasoning is much more complex than the listing of possibilities for an assertion, and so reasoners should tend to rely on System 1′s mental models. A special case is inferences that have erroneous mental models that fully explicit models correct. They should create compelling fallacies that have the force of cognitive illusions. So, for all three sorts of modal premises (alethic, deontic, and epistemic), the theory makes:

> **Prediction 4.** Inferences that call only for mental models should be easier than those calling for fully explicit models, and those premises that have erroneous mental models should create illusory inferences.

The predicted illusions provide a crucial test for the use of mental models, because no other theory of reasoning predicts them. Experiments have corroborated that individuals succumb to modal illusions (Goldvarg & Johnson-Laird, 2000; Khemlani & Johnson-Laird, 2009). Perhaps the simplest examples are those in which the task is to evaluate whether it is possible for two assertions both to be true, as in the following problem based on exclusive disjunctions:

69. Either the pie is on the table or else the cake is on the table.
    Either the pie is on the table or else the cake isn't on the table.
    Could both of these assertions be true at the same time?

A positive answer to the question depends on the two assertions having a possibility in common. Most participants in an experiment responded 'Yes' (Johnson-Laird, Lotstein, & Byrne, 2012). The mental models of the two premises have in common that the pie is on the table. But, fully explicit models show that this evaluation is an illusion: the two premises refer to no possibility in common (see C1 in Appendix C). Inferences that have erroneous mental models do create cognitive illusions (Bucciarelli & Johnson-Laird, 2005; Khemlani & Johnson-Laird, 2009). Their occurrence is a sign of inferences based on mental models.

5.1. Alethic inferences

Alethic reasoning occurs in the assessment of consequences in order to evaluate whether they are necessary, possible, or impossible. Because of the interdefinability of alethic modalities, their models could represent what is necessary rather than what is possible. But, the theory postulates that models represent possibilities. One consequence is a predicted interaction. A possibility can hold in a single model of premises but a necessity has to hold in all of them, whereas the negation of a necessity can hold in a single model of the premises but the negation of a possibility has to hold in all of them. The number of models needed for an inference predicts its difficulty, and so the theory yields the interaction:

> **Prediction 5.** Given premises that yield multiple models, individuals should be faster and more accurate to infer a possibility than a necessity, but the opposite should occur for their denials.

A test of this interaction used one-on-one games of basketball in which only two can play (Bell & Johnson-Laird, 1998). There are four putative players, *A, B, C, D*, and each player 'is in' a game or 'is out' of it. A typical problem is:

70. If A is in then B is in.
    If C is in then D is out
    Can B be in?

Given that two must play, mental models of the possible games to which the premises refer are:

A	B		
	B	C	
	B		D

The initial mental model answers the question, and so, as the interaction predicts, the participants were faster and more accurate in answering 'Yes', than when the question was:

71. Must B be in?

Its right answer for the right reason depends on all three models above. For inferences in which the correct answer is 'No,' the experiment used a dual of the previous problem, interchanging 'in' and 'out'. And, as the interaction predicts, the participants were faster and more accurate in responding 'No' to 'Must B be in? than to 'Can B be in?'. The study, which examined various sorts of problem including those based on disjunctions, corroborated the interaction. So, it follows that models do represent possibilities even in a domain in which they could instead represent necessities.

The alethic assessment of inferences illustrates the role of relations between conjunctions of default possibilities – those for the premises and those for the conclusion. The premises and conclusion may have no semantic relation to one another, e.g.:

72. It is snowing or it is freezing.
    Therefore, the number is prime.

The two sets of possibilities are disjoint, and so the conclusion is consistent with the premise, but it would be misleading to assert that it follows as a possibility from the premise. In contrast, given that the premises and conclusion have a semantic relation and that they both yield models of possibilities, the alethic assessment of the conclusion is simple. It depends on the conjunction of the models for the premises and the models for the conclusion. If this conjunction of the two sets yields the same models as those for the conclusion, then the conclusion is true for all the cases to which the premises refer, and it follows as *necessary*, e.g.:

73. It's cold and either it's snowing or else it's raining.
    Therefore, necessarily, either it's snowing or else it's raining.

Because the conclusion is necessary, the truth of the premises guarantees the truth of 'either it's snowing or else it's raining' – it's a valid inference. Otherwise, provided that the conjunction does not yield the null model representing a self-contradiction (see Appendix C), the conclusion is true in at least one case to which the premises refer, and so it follows as a *possibility*, e.g.:

74. It's hot.
    Therefore, possibly, it's hot or cloudy, or both.

A corollary is that the truth of the premises guarantees the truth of the conclusion: 'it is possible that it's hot or cloudy, or both' – it's a valid inference. If the conjunction does yield the null model, then the conclusion isn't true for any case to which the premises refer, and so given the premises, it follows as *impossible*. A consequence of these principles is:

> **Prediction 6.** Individuals have the competence to draw conclusions that are necessary, possible, or impossible.

The preceding study of inferences about basketball games showed that they drew all of these different sorts of modal conclusion (see also Evans, Handley, Harper, & Johnson-Laird, 1999).

5.2. Deontic inferences

Consider this problem:

75. Having children obligates you to take care of them. (C obligates T.)
    To take care of children prohibits you from leaving them. (T prohibits L.)
    What follows?

The mental models of the deontic possibilities are, in abbreviation:

children	taking care	¬ leaving
	…	

It follows that:

76. Having children prohibits you from leaving them. (C prohibits L.)

A contrasting problem is one of the sort:

77. Having children permits you to take care of them.
    Taking care of children prohibits you from leaving them.
    What follows?

The mental models of these premises have only one model of all three elements:

children	taking care	¬ leaving

So, individuals should tend to infer the same conclusion as before (76). But, as fully explicit models show, having children can occur without your taking care of them, e.g., you employ a baby-sitter, and in this case you can leave them. So, according to the premises, having children doesn't prohibit you from leaving them. But, reasoners should tend to base their inferences on mental models of the premises rather than on fully explicit models. An experiment contrasting inferences of the two sorts corroborated the prediction (Bucciarelli & Johnson-Laird, 2005). For example, 95% of participants formulated the valid conclusion for the inference (75), whereas only 25% did so for inference (77) – the majority inferred the prohibitory conclusion in (76) or the common response to multiple-model problems that nothing follows.

5.3. Epistemic inferences

In daily life, inferences from premises referring to epistemic possibilities are ubiquitous. Compound assertions such as conditionals and disjunctions about can imply such conclusions. Because the theory is based on conjunctions of default possibilities, it yields:

> **Prediction 7.** Reasoners should reject epistemic inferences as guaranteeing the truth of their conclusions if the possibilities to which the premises refer support some but not all the possibilities to which the conclusion refers: the conclusion is only possible.

Studies have shown, for instance, that participants reject inferences from a categorical assertion to a disjunctive conclusion (Orenes & Johnson-Laird, 2012), as in:

78. It's freezing.
    So, it's freezing or it's snowing, or both.

The premise refers to only one of the possibilities to which the conclusion refers, and so the conclusion is only possible. Likewise, this study also showed that reasoners reject the well-known 'paradoxes' of material implication, such as: *C*; therefore, *If A then C*. The premise does not refer, for instance, to the possibility of *A and C* to which the conclusion refers.

Another test is that inferences from an exclusive disjunction to an inclusive disjunction, as in:

79. It's freezing or it's snowing, but not both.
    Therefore, it's freezing or it's snowing, or both.

should also be rejected as guaranteeing the truth of the conclusion. They are rejected, and the mismatch between 'but not both' and 'or both' cannot explain the results (see Hinterecker et al., 2016). The premise does not imply one of the possibilities to which the conclusion refers – it's freezing and snowing. Yet, inferences (78) and (79) are valid in sentential logic and in probabilistic logic too (Adams, 1998).

So far, the theory has made predictions that prior studies have corroborated. We now turn to three predictions that the theory makes, and that inspired new studies to test them (Ragni & Johnson-Laird, 2019).

The model theory postulates that an assertion of the sort *it is possible that A* presupposes that *it is possible that not-A* (see 42). So, the theory yields the consequence:

> **Prediction 8.** Reasoners should accept the inference: Possibly A; therefore, possibly not-A, where A is simple or compound, because the premise presupposes the conclusion. They should likewise accept the converse inference.

An experiment showed that individuals tended to accept such inferences, e.g.:

80. It is possible that it will rain.
    Therefore, it's possible that it won't rain.

and the converse inference (Ragni & Johnson-Laird, 2018). A revealing example, which we owe to a reviewer, is:

81. It is possible that Trump will not run the next NYC marathon in less than 2 s.
    So, it is possible that Trump will run the next NYC marathon in less than 2 s.

The inference fails, because its premise presupposes its conclusion, but the latter is false. Assertions whose presuppositions are false cannot be true, and so the premise is undercut by the falsity of its own presupposition – the price for a bizarre weak claim.

Mental models are parsimonious: they do not represent what is false. Parsimony likewise yields this principle:

82. Mental models represent as few distinct possibilities as is feasible, and so separate possibilities are condensed into one unless they are inconsistent.

Suppose, as in our opening example (3), you learn that that Pat may be married, and then you learn that Viv may not be. Parsimony predicts that you think of a single possibility with its presupposition of falsity:

Pat married	¬ Viv married
	…

As a consequence, you accept the conclusion:

83. Maybe Pat is married and Viv isn't.

The inference is invalid in any normal modal logic, because one possibility could be inconsistent with the other. Hence, the theory yields:

Prediction 9. System 1 should condense possibilities into a single default possibility unless they are inconsistent, and so individuals should accept the corresponding inferences.

Knowledge can modulate the interpretation of assertions (see 3.2.). It blocks the conjunction of two possibilities into one if they are inconsistent with one another, e.g.:

84. Pat may be married and Pat may be single

cannot be condensed into:

85. Maybe Pat is married and single.

Experiments have confirmed that reasoners condense consistent possibilities into one but not inconsistent ones (Ragni & Johnson-Laird, 2019).

Because epistemic necessities refer to necessary conditions for other states of affairs, the theory yields:

Prediction 10. A single assertion of an epistemic possibility in premises should elicit a conclusion about a possibility, whereas a single assertion of a necessity states a necessary condition for another state of affairs, and so it should not tend to elicit a conclusion about a necessity.

For example, in the following sort of inference:

86. It is possible that Alex is in Erie.
If Alex is in Erie than Eddy is in Fremont.
What, if anything, follows?

the predicted conclusion is:

87. It is possible that Eddy is in Fremont.

This conclusion is invalid in all normal modal logics. In contrast, the inference:

88. It is necessary that Alex is in Erie.
If Alex is in Erie than Eddy is in Fremont.
What, if anything, follows?

should not elicit the conclusion that it is necessary that Eddy is in Fremont, because the first premise states a necessary condition for some other state of affairs. The results of two experiments, one in which the participants drew their own conclusions and the other in which they evaluated given conclusions, corroborated these predictions (Ragni & Johnson-Laird, 2019).

5.4. A synopsis of predictions

We have now completed the account of the model theory. We have boiled it down to ten predictions about the modal interpretations of assertions, and about inferences from them. Table 3 summarizes these predictions and cites examples of studies that support them.

6. General discussion

Ask a psychologist how people reason, and the answer you tend to get is that they use logical rules (e.g., Piaget, 1957; Rips, 1994). So, they should use modal logics to reason about possibilities. But, in contrast to all normal modal logics, people are happy to infer that nothing follows from certain premises, to withdraw conclusions that evidence contradicts, not to infer anything from a contradiction, and to deal with possibilities of different sorts within a single assertion (e.g., 8). But, if people do not rely on normal modal logics, then how do they reason?

The answer seems to be that they rely on models of possibilities. Earlier research (see Section 2) had shown that "possible worlds" were too vast to make a plausible psychological semantics for possibilities, and that no other feasible account existed. The present article therefore argued (in Section 3) for a finitary semantics founded on the ability to categorize any situation into a small number of exhaustive alternatives – each of which could be realized in an infinite number of different ways. Any subset of the finite alternatives is a possibility. This semantics underlies the three main modalities of everyday life: alethic, deontic and epistemic. Alethic knowledge concerns relations between two sets of possibilities, and includes necessary and possible inferences, and causal relations. Deontic knowledge concerns social relations governing obligations and permissions, which speech acts themselves can create. And knowledge of the world governs epistemic interpretations varying on a scale of degrees of belief from impossible to certain (akin to subjective probabilities). The theory postulates (in Section 4) that possibilities are represented in models, which are iconic representations of what is common to their infinite realizations. Mental models represent what is true for an intuitive System 1, which considers them one at a time. In simple tasks, they can be fleshed out into the fully explicit models of a more powerful deliberative System 2. As evidence bears out, models represent possibilities rather than necessities, and they distinguish among alethic *causes*, *prevents*, and *allows*, deontic *obligates*, *prohibits, permits*; and the epistemic scale of degrees of belief from *impossible* to *certain*. Evidence also shows that compound assertions, such as conditionals and disjunctions, do not have the truth-functional meanings of logic in which their truth or falsity depends only on the truth or falsity of the clauses they connect (see Appendix A). Instead, they refer to conjunctions of possibilities that hold in default of knowledge to the contrary.

Modal reasoning depends on conjoining the models of premises (Section 5). We described a procedure in Appendix C to enable readers to construct their own mental models and fully explicit models. Mental models yield systematic fallacies for all three sorts of modality, which are compelling enough to be cognitive illusions. In principle, fully explicit models can correct such errors, but inferences can impose too great a load on working memory for reasoners to use these models.

The possibility of a proposition presupposes the possibility of the proposition's negation. So, in corroboration of the theory's new predictions, people infer from the epistemic possibility that it will rain, the possibility that it won't, and vice versa. Likewise, they tend to condense consistent epistemic possibilities into one. The three opening examples (1–3) in this article are all valid in the model theory, but not in any normal modal logic. Indeed, as we have illustrated, reasoners make certain inferences that are invalid in all such logics, and fail to make other inferences that are valid in all of them. Instead, they follow the model theory. No other viable psychological theory of modal reasoning

Table 3
The model theory's ten principal predictions about the interpretations of modals and reasoning from them, with citations to examples of their experimental corroborations.

Prediction	Corroboratory study
The interpretations of modals	
1. Compounds such as disjunctions and conditionals imply conjunctions of default possibilities.	Hinterecker et al. (2016).
2. Sets of alethic possibilities discriminate the meanings of *causes*, *prevents*, and *allows*.	Khemlani et al. (2014).
3. Sets of deontic possibilities discriminate the meanings of *obligates, prohibits,* and *permits*.	Bucciarelli and Johnson-Laird (2005).
Reasoning with modals	
4. Inferences from mental models of any sort of modal are easier than from fully explicit models, and illusions occur.	Khemlani and Johnson-Laird (2009).
5. Alethic inferences of possibility are easier than those of necessity, but their denials switch in ease.	Bell and Johnson-Laird (1998).
6. Alethic inferences are necessary, possible, or impossible.	Evans et al. (1999).
7. Epistemic conclusions are not guaranteed to be true if premises fail to refer to all the conclusion's possibilities.	Orenes and Johnson-Laird (2012).
8. Epistemic *possibly A* presupposes *possibly not-A*.	Ragni and Johnson-Laird (2019).
9. Inferences condense epistemic possibilities into one unless they are inconsistent.	
10. Epistemic premises about a possibility should yield inferences of possibilities, whereas premises about epistemic necessity should not yield inferences of necessities.	

appears to exist at present. There are, however, three potential rival accounts.

First, theorists could try to defend modal logic by adding pragmatic assumptions of the sort proposed in Grice (1989). The model theory predicts the following sort of inference, which people make (see 50–51) even though it is invalid in all normal modal logics:

89. The flaw is in the software or it is in the cable, or both.

So, it is possible that the flaw is in the software.

Instead of treating these default possibilities as part of the meaning of disjunctions, this approach would posit that they are 'conversational implicatures' (Grice, 1989). Which are not valid inferences, because speakers can cancel them without contradiction. But, suppose a speaker cancels both of the possibilities to which the disjunction in (89) refers:

90. The flaw is in the software or it is in the cable, or both, and in fact the flaw is not in the software and not in the cable.

The assertion is now self-contradictory. Indeed, the model theory implies that to deny both possibilities is to deny the disjunction. An approach based on conversational implicatures does not seem to be viable, and in any case it fails to explain several other predictions (see Table 3).

A second potential account is the 'new paradigm' in which the probability calculus is supposed to replace logic in theories of reasoning (see, e.g., Evans, 2012; Fugard, Pfeifer, Mayerhofer, & Kleiter, 2011; Oaksford & Chater, 2007; Over, 2009). Its proponents criticized the model theory on logical grounds (Baratgin et al., 2015; Cruz, Over, & Oaksford, 2017), but defenders of the model theory have replied that it should be judged on its own terms (e.g., Hinterecker, Knauff, & Johnson-Laird, 2018), and that unlike probabilistic theories it explains reasoning about possibilities. At the heart of the probabilistic approach is the Equation in which the probability of a conditional, *if A then B*, equals the conditional probability of *B* given *A* (Adams, 1998). This equation is also a consequence of the model theory's possibilities for conditionals (see 4.2.3.). The assessment of probabilistic theories is difficult because they have no algorithmic account of mental processes. Yet, they do face critical problems. One is that people do not know how to calculate probabilities, and so their subadditive estimates can be gross, yielding probabilities that sum to much more than 100% (see 4.2.3.). Probabilistic theories need to accept this limitation rather than to claim that people make numerical estimates consistent with the probability calculus. Likewise, these theories accept the concept of probabilistic validity in which no uniform assignment of probabilities yields a lower probability for the conclusion than for the premises (Adams, 1998). However, people reject certain inferences that are probabilistically valid, e.g., those of the sort, *A*; therefore, *A or B or both* (see Section 5.3). They also accept certain inferences that are not probabilistically valid (e.g., 50–1). The standard probability calculus cannot assign a probability to an assertion about possibility, and so theorists introduce special assumptions to do so (see Appendix B). Even so, reasoners' own judgments of probabilities show that they infer conclusions about possibilities that are less probable than their premises (Hinterecker et al., 2016). And probabilistic theories cannot distinguish between certainty and necessity, because both have a probability of 100%. So, an assertion such as:

91. It is possible for Hillary to run again but impossible that she will has no probabilistic paraphrase. And a speech act creating an obligation – a deontic necessity, such as:

92. You must leave now cannot be formulated in a sentence only about a probability. The obligation means that you must leave, not that your probability of doing so is 100%. Indeed, you may violate your obligation and fail to leave.

Could a theory of reasoning nonetheless treat obligations as implicitly probable, or as a subset of the broader concept of 'normality' (Bear & Knobe, 2017)? The problem with such suggestions, which we owe to a reviewer, is that implicit probabilities would imply that deontic states are scalar, but they are not. You either have permission to leave or you don't. If an event is normal then it is possible, but the converse claim is false. So, you can create an obligation in an explicit denial of normality, e.g.:

93. I know it's not normal, but I hereby forbid you from leaving the party.

It is not a self-contradiction, but it would be if normality included all obligations.

A third potential alternative is due Phillips and Knobe (2018). They argue that certain phenomena are common to thoughts about possibilities, freedom to act, the selection of a cause from relevant factors, and counterfactual reasoning. What is relevant, they say, are physics, morality and probability, which people can take into account in a unified representation of modality. People grasp that an assertion, such as:

94. You can't do that

could refer to a physical impossibility, a moral impossibility, a probability, or to an admixture of them. They therefore adopt Kratzer (1981) approach of a single underlying formal semantics based on possible worlds (see Appendix B). They recognize that people focus on only certain possibilities, and so they envisage a function that maps an actual situation onto a set of possibilities, and then a further function that puts them into an order of likelihood. The result is that the set of possibilities excludes improbabilities and violations of physics or morality. But, consider such a set of possibilities for a coin toss. The coin can vary in dimensions, inscriptions, and trajectories. So, there remains a potential infinity of different ways in which the coin toss occurs. We have arrived back at possible worlds that are too vast to be a plausible semantics for modality. The model theory, in contrast, has a finite semantics, an algorithm for reasoning, and allows that possibilities concern more than physics and morals, e.g., deontic modalities can refer to rules of games and social conventions (Bucciarelli, Khemlani, & Johnson-Laird, 2008).

Many questions remain about the model theory and call for new lines of research. One such line is the theory's prediction about such inferences as:

95. You may have the soup or the salad, or both.
Therefore, you may have the soup.

They are valid in the model theory, because the premise refers to a conjunction of possibilities, which includes the one corresponding to the conclusion. Logicians refer to such inferences as 'paradoxes' of free choice (see Appendix B), because they are invalid in normal modal logics. However, the model theory makes the same prediction for epistemic inferences, such as:

96. He may have had the soup or the salad, or both.
Therefore, he may have had the soup.

We are investigating these inferences.

Another line of research needs to consider assertions, such as:

97. You ought to see *Hamlet*

This sort of assertion is sometimes classified as deontic (e.g., Lassiter, 2017, Ch. 8), but it lacks the essential characteristic of functioning as a speech act to create a modal state. It expresses an opinion – a recommendation based on the underlying semantics of two alternatives, and the speaker's view that the one in which you see *Hamlet* is better than the one in which you don't. It is an open question whether the model theory can accommodate such modalities.

If possibilities are the foundation of human reasoning, then they and their cognates should underlie quantified assertions, such as:

98. All the artists in the show were Cubists.

They appear to do so, and to yield inferences, such as:

99. Braque was a Cubist, and so he may be in the show.

Predicate modal logics can analyze such inferences (see, e.g., Girle, 2009), but they inherit all the problems that we have identified for normal sentential modal logics. Yet, models of quantified assertions seem to accommodate modal claims in a way that yields such inferences (e.g., Khemlani, Lotstein, Trafton, & Johnson-Laird, 2015).

Yet another line of research concerns the extension of the model theory to deal with pre-linguistic infants. They can reason about the identity of a toy, making inferences of the sort: *It's A or B*; *it isn't A*; so, *it's B*. They look longer – a sign of surprise – if they discover that the toy isn't *B* (Cesana-Arlotti et al., 2018). The children seem able to represent the two possibilities, and to infer that when one is eliminated, the other possibility is the case. But, it could be that they have an analogous formal rule of inference. One finding that supports a potential extension of the model theory is that older children from different cultures are able to prepare for alternative future possibilities (Redshaw et al., 2019). In sum, although the model theory has empirical support, several lines of research need to be pursued in order to test its potential extensions.

7. Conclusions

The model theory postulates that the meanings of *possible* and its cognates refer to small finite numbers of alternatives. Knowledge can interpret them as alethic relations, which include allowing and causing. It can interpret them as deontic permissions or obligations, which speech acts can create. And it can interpret them as epistemic possibilities or certainties, which are probabilities with or without numbers. The meaning of 'or', 'if … then …', and other sentential connectives, refers to conjunctions of models of possibilities that hold in default of knowledge to the contrary. All these interpretations have mental models (in system 1) supporting intuitive inferences, which can be fallacious. Deliberation can flesh out mental models into fully explicit ones (in system 2), which correct the fallacies. The patterns of modal reasoning corroborated the theory's predictions, including the acceptance of inferences invalid in all normal modal logics, and the rejection of inferences valid in all of them. Because the model theory embodies finite limitations on representations, it is broader and perhaps more plausible than other accounts.

Can possibilities be the foundation of human reasoning? As we know to our cost, the hypothesis is too incredible for many theorists to believe. Yet, its predictions are borne out so far, both in pre-existing evidence and in new experiments that the theory inspired. There may be infinitely many possible worlds whose denizens reason on the basis of normal modal logics. We have discovered that the real world is not amongst them. Some other as yet unknown mechanism may underlie modal reasoning, but it appears to depend on finitary models of possibilities. Perhaps, as Sherlock Holmes remarked about a different matter, it is more than possible.

Acknowledgements

This research was made possible by a grant to the first author from the DFG in proposal RA 1934/3-1 and RA 1934/4-1. The article has had a long gestation, and we are indebted to many individuals. We thank Sunny Khemlani for advice and a critical reading of an earlier draft. We thank colleagues for their help and ideas: Monica Bucciarelli, Ruth Byrne, Geoff Goodwin, Thomas Hinterecker, Marie Jacobs, Markus Knauff, Philipp Koralus, Robert Mackiewicz, Gorka Navarrete, Isabelle Orenes, Cristina Quelhas, and Carlos Santamaría. We thank experts on semantics for their advice: Adele Goldberg, Dan Lassiter, Barbara Partee, Paul Portner, John Sowa, and Mark Steedman. We thank Ori Friedman, an active action editor, four anonymous reviewers, and critics of the model theory for their trenchant critiques: Nicole Cruz, Jonathan Evans, Gernot Kleiter, Mike Oaksford, Klaus Oberauer, David Over, and Guy Politzer. Finally, we thank innocent bystanders for their insightful comments: Ernie Davis, Maya Bar-Hillel, Greg Murphy, Keith Oatley, Dan Osherson, Bob Rehder, Dan Sperber, and participants in the London Reasoning Workshop and the Concepts and Categories seminar at New York University.

Appendix A. Logic and modality

Modal logic is the logic of possibility and necessity, and goes back to Aristotle's *Prior Analytics* (Barnes, 1984). Not all logicians accept it. Frege (1879/1967) in formulating modern logic rejected it, because possibilities depend on knowledge. His contemporary, Peirce, who formulated an independent version of modern logic, embraced modality (Peirce, 1931-1958, Vol. 5, paragraph 510, see also Sowa, 2018). Logicians went on to explore alternative sets of axioms yielding different modal logics (e.g., Lewis & Langford, 1932). They also distinguished between 'proof' theory, which is a formal system of rules for deriving proofs, and 'semantic' theory, which determines the meanings of logical terms and validity (Tarski, 1936/1956).

The simplest normal modal logic, system K (for Kripke), has both a proof theory and a semantic theory. The system is *sound* because all its proofs are valid inferences; and it is *complete* because all its valid inferences have proofs. Other logics are not complete, such as the logic of arithmetic. In normal modal logics such as K, sentences refer to possible worlds. So, a sentence, such as:

A1. It is possible that it is raining in Manhattan.

is true if, in turn, it is true in at least one relevant possible world that it is raining in Manhattan.

So, in general:

A2. $\Diamond$A in world w

which denotes *possibly A holds in world w,* is true if sentence *A* is true in at least one possible world relevant to *w*. Each possible world relevant to the real world is therefore vast, because each sentence about the real world is either true or false in it. The sentence:

A3. $\Box$A in world w

which denotes *necessarily A holds in world w,* is true if sentence *A* is true in all possible worlds relevant to *w*. Relevance, or 'accessibility' as it is often known, has by design different definitions. A relevant world for *w* can refer to a world that is conceivable in *w* given knowledge (for epistemic logics), to one that occurs in a future from *w* (for temporal logics), or to one that has the same principles for permissions and obligations as *w* (for deontic logics). Modal logics burgeoned after the discovery that different definitions of relevance correspond to different axioms in proof theory (e.g., Kripke, 1963). Interest in them has spilled over into artificial intelligence (Meyer & van der Hoek, 1995) and into systems of nonmonotonic reasoning for modality (McDermott, 1982).

All normal modal logics contain the sentential calculus, which deals with connectives akin to 'if', 'or', and 'and'. Their meanings are treated as truth-functional in that whether sentences containing them are true or false depends only on the truth or falsity of the clauses they connect. These meanings can be spelt out in a 'truth table' showing the truth or falsity of a compound sentence for each of the cases depending on the affirmation or negation of each of its clauses, i.e., its *partition*. So, an inclusive disjunction of the sort, *A or B, or both*, has the following truth table, where *not-A* denotes the negation of *A*:

Case	Truth value of *A or B, or both*
A and B	True
A and not-B	True
not-A and B	True
not-A and not-B	False

Likewise, in sentential logic, *If A then B* corresponds to the *material conditional*, symbolized as: A → B. Its truth table stipulates that it is true in three cases in its partition: *A and B*, *Not A and B*, or *Not A and not B*, and that it is false only in the case of *A and not B*.

The proof theories for all normal modal logics add to axioms (or equivalent rules of inference) for the sentential calculus those for the modal operators, *possible* and *necessary*. These logics have names, such as "system T". Their semantic theory, however, cannot be based on truth tables; otherwise, $\Box A$ would be equivalent to *A*. Instead, it is usually based on possible worlds and on a set of assumptions about relevance. Which inferences are valid therefore differ from one system to another. System K, the simplest modal logic, makes no assumptions about relevance. It underlies an infinite family of normal modal logics. Like its descendants, system K treats the two modal operators as interdefinable using negation:

A4. A is possible if and only if not-A isn't necessary.
A is necessary if and only if not-A isn't possible.

System K posits the following rule of inference, or its equivalent axiom:

A5. $\Box$(A → B). Therefore, ($\Box$A → $\Box$B).

In other words, *the necessity of if A then B* implies that *if A is necessary then B is necessary*. Other equivalent ways exist to formulate this axiom. System K, however, is an implausible candidate for human reasoning, because in daily life a necessary conclusion is the case:

A6. $\Box$A → A

This principle cannot be proved in system K, so it is added as an axiom to K to create system T. The axiom (A6) corresponds to the semantic assumption that each possible world is relevant to itself, i.e., the relation of relevance is reflexive. Hence, what is necessary in world *w* holds in world *w*.

Various deontic logics exist (e.g., van Fraasen, 1973; Lewis, 2000). A principle that they eschew is that what is obligatory occurs – people do not always do what they ought to do, and so these logics exclude axiom T (in A6). But, they do include the axiom that any obligatory action is permissible:

A7. $\Box A \rightarrow \Diamond A$ – *if A is obligatory then A is permissible.*

This principle added to system K, but without axiom T, yields deontic system D.

In colloquial English, a sentence can contain several modal operators, e.g.:

A8. Perhaps it's possible that it may rain.

It could be symbolized as:

A9. $\Diamond\Diamond\Diamond$rains.

But, suppose that what is possibly possible is possible. This principle is embodied in system S5,

which is based on T, and is therefore in K's family. S5 collapses strings of modal operators into the final one in a string, e.g.:

A10. $\Diamond\Diamond\Diamond A \rightarrow \Diamond A$

As a consequence, S5 has just six modalities: $\Box A$, $\Diamond A$, A, and their respective negations, where A denotes a factual sentence, which is either true or false in the world to which it refers, irrespective of other possible worlds. A logic could allow that three modal operators such as $\Diamond\Diamond\Diamond A$ imply two $\Diamond\Diamond A$, but not that two modal operators imply one $\Diamond A$. This principle can be extended upwards, so that n operators imply n − 1 operators, but no further, where n is any natural number, e.g., a string of 15 operators in a sentence imply one with 14 operators, but no further reductions in their number are allowed. It follows that there is a countable infinity of distinct normal modal logics.

Appendix B. Modality in linguistics

The most influential studies are due to Kratzer, who analyzed German modal auxiliary verbs, for which she proposed a common underlying semantics of 'possible worlds' for their different interpretations (Kratzer, 1977, 1981, 1991, 2012). She argued for a common invariable kernel in the meaning of 'must' whether it is used to make a deontic, epistemic, or other sort of claim (see also Johnson-Laird, 1978). This commonality can be brought out in paraphrases referring to the conversational context that constrains interpretations, e.g.:

B1. *In view of the tribal duties*, children must learn the name of their ancestors.

The italicized phrase makes explicit the context of the subsequent clause. Kratzer postulates that the proposition that a sentence expresses in its context is the set of possible worlds in which it is true. Knowledge is itself a set of propositions, and so it too can be captured in possible worlds: a function assigns to each possible world those propositions that are known in that world (cf. Hintikka, 1962).

In later studies, Kratzer (2012) noted the subtle effects of different conversational backgrounds on the interpretation of epistemic 'must'. Some assertions, but not all, commit the speaker to the truth of the proposition within the scope of the modal (von Fintel & Gillies, 2010), e.g.:

B2. Given the article in the Hampshire Gazette, Higgins must have been re-elected.

This issue has generated a large literature, relating to the use of 'must' to reflect an inferred proposition, as in the previous example, or else its use to refer to direct evidence, e.g.:

B3. You must have a cold. Your nose is dripping.

But, necessity need not make a factual claim: *you must be the plumber* makes a weaker claim than *you are the plumber* (Karttunen, 1972; Steedman, 1977; but cf. von Fintel & Gillies, 2010).

Kratzer (2012) crucial insight is that epistemic possibilities are on scale, i.e., they are gradable, and so a possibility can be 'slight' or 'good', and it can even be 'probable' (see also White, 1975). Not all modal terms can be qualified in this way (Portner, 2009). The standard probability calculus cannot assign probabilities to possibilities. So, in an earlier version of her article, she treated these grades as orderings on relevant possible worlds; in the later version, she showed – following Lewis (1981) – how numerical probabilities ranging from 0 to 1 can emerge from a grading of possible worlds. She also points out the existence of non-numerical, though graded, probabilities (echoing Keynes, 1921). To paraphrase her possible-worlds semantics (Kratzer, 2012, p. 40):

B4. A proposition is a possibility in a world, *w*, with respect to its conversational background and a source for grading worlds if, and only if, its negation is not necessary in *w*.

Hence, modal propositions depend on three arguments: a sentence, its conversational context expressible in the sentence itself, and the source of its grading.

Graded modals have a magisterial treatment in Lassiter (2017). He argues that scales can have upper or lower bounds, or both, and that rank order, interval, and ratio scales, have increasing arithmetical properties in ways analyzed in psychology (e.g., Stevens, 1946; Krantz, Luce, Suppes, & Tversky, 1971). Hence, the scale for certain deontic modals has neither upper nor lower bounds, and is a sort of interval scale, whereas the scale for epistemic modals has both upper and lower bounds and is a ratio scale. He introduces a theory of the translation between qualitative and quantitative scales. (For an independent algorithm that translates qualitative into quantitative probabilities, see Khemlani et al., 2015). An assertion such as:

B5. Rain is exactly twice as likely as snow

shows that 'likelihood' has a ratio scale akin to the ordinary numerical scale of probability.

To treat epistemic possibilities as scalar is either to repudiate its traditional 'all-or-nothing' interpretation, as Lassiter does, or else to argue that this latter interpretation can be coerced into a scalar one. The evidence for its scalar nature is the validity of inferences, such as the one in example (2) at the start of our paper, and the similar lower bound of impossibility for both possibility and probability. So, Lassiter argues, epistemic possibilities and probabilities share a common scale.

Lassiter (2017, Ch. 3) repudiates possible-world semantics for the scale: it does not work for 'probable'. For example, Kratzer's semantics for epistemic modals implies the validity of the following dubious inference:

B6. It is as likely that Bill will do action A as action B.
It is as likely that Bill will do action A as action C.
Therefore, it is as likely that Bill will do action A as action B or action C.

Lassiter also argues against a unified semantics across different interpretations of modals, because deontic and epistemic necessities differ in their logical properties (see the difference between systems T and D in Appendix A.). Lassiter does not propose a semantics for modals, but he leans towards a probabilistic account (e.g., Oaksford & Chater, 2007). He carries out experiments to test his hypotheses, and, unlike theorists who rely on possible worlds, he argues that the language of modality has a close relation to the mental representations that underlie reasoning (Lassiter, 2017, p. 259).

One other development in linguistics presages the model theory. A long-standing puzzle in deontic logics is so-called 'free choice' inferences from permissions, such as:

B7. You may have the soup or the salad, or both.
Therefore, you may have the soup.

In deontic logics, this inference is invalid, whereas it is converse is valid (Cariani, 2017; von Wright, 1969). Zimmermann (2000) proposed a solution: the meanings of disjunctions are equivalent to conjunctions of alternative possibilities. So, the premise in (B7) means in effect:

B8. You may have the soup *and* you may have the salad *and* you may have both.Geurts (2005) amended and extended this account so it applied to deictic and epistemic disjunctions and conditionals. But, both these accounts have a serious problem with negation. A disjunction without modal clauses, such as:

B9. Gray is professor or a judge or both

implies a conjunction of three possibilities. But, its negation means that Gray is neither a professor nor a judge, and naive individuals concur (Khemlani et al., 2014). But, the negation of a conjunction of the three possibilities to which (B9) is supposed to refer is as follows:

B10. It is not the case that Gray may be a professor and that he may be a judge and that
he may be both.

It means that at least one of the possibilities is false, and not necessarily that all three of them are. Geurts tries to explain the problem away by distinguishing between *speaker's content*, which commits the speaker 'to an opinion, a desire, a course of action, or whatever,' and *factual content*, which demarcates 'the domain within which such commitments holds.' Speaker's content treats disjunctions as conjunctions of possibilities whereas factual content treats them as logical disjunctions. The proposal is at best ad hoc. For instance, the negation in the following assertion:

B11. In my opinion, it's not true that Gray is either a professor or a judge or both.

should have speaker's content, and yet it calls for the logical interpretation of factual content. In following in Zimmermann's and Geurts's footsteps, the model theory solves the problem of negation without an appeal to two sorts of content (see 3.3.).

Appendix C. The construction of models

The model theory postulates that a parser builds up models in a compositional way in which each rule in the grammar has a semantic principle that constructs interpretations, depending ultimately on the meanings of words or morphemes. Mental models, which underlie intuitive reasoning (in system 1), take into account the principle of truth (56): each mental model represents only those clauses in a premise that are true in the model. Fully explicit models, which underlie deliberative reasoning (in system 2), represent the status of each clause in the premise in all the models. All sentential connectives have meanings based only on only conjunction and negation. For instance, an exclusive disjunction *A or else B*, in which both clauses cannot be true, is defined to have the models shown in Table 2, where capital letters, such as, *A* and *B,* denote simple clauses or compound clauses that contain sentential connectives, and *not-A* denotes the negation of *A*. For fully explicit models, the meaning of the exclusive disjunction above is therefore:

A and not-B
Not-A and B

We now explain informally how to form the conjunction, *and*, of two sets of models and how to form the negation, *not*, of a set of models.

The conjunction of models does double duty, because it is used to interpret connectives, and to combine the models of separate premises. If one element in a model contradicts its occurrence in the other model, there is no need to conjoin the two models, because they contradict one another. The same applies if one model contains an element missing from the other, which is in the set containing this other model. The missing element, a, is then equivalent to its negation, ¬ a. This procedure should become clearer in an example. The following premises are both exclusive disjunctions:

C1. Either the apple is on the table or else the banana is on the table.
Either the apple is on the table or else the banana is not on the table.

The process of making a conjunction of their respective mental models has two steps:-

1. Write down the mental models for both premises (see Table 2 in the main text). The models for the first premise above are in abbreviation:

a	
	b

and those for the second premise are:

a	
	¬ b

where '¬' denotes negation.

2. Make the pairwise conjunctions of the two sets of models, bearing in mind the treatment of missing elements and contradictions. There are four pairwise conjunctions in our example:

First premise			Second premise		
a		and	a		yields a
a		and		¬ b	do not conjoin (as ¬ b is in a model missing a)
	b	and	a		do not conjoin (as is b in a model missing a)
	b	and		¬ b	do not conjoin (as b contradicts ¬ b)

The resulting model is therefore:

a

The process for fully explicit models is similar:-

1. Write down the fully explicit models for each premise (see Table 2 in the main text). So, they are, respectively:

First premise			Second premise	
a	¬ b	and	a	b
¬ a	b		¬ a	¬ b

2. Make the pairwise conjunctions of the two sets of models, bearing in mind the treatment of contradictory elements. There is no need to conjoin such models, because the result would be nil, the null model. It is represented only if the conjunction of two sets yields no other sort of model:

First premise			Second premise		
a	¬ b	and	a	b	do not conjoin (as ¬ b contradicts b)
a	¬ b	and	¬ a	¬ b	do not conjoin (as a contradicts ¬ a)
¬ a	b	and	a	b	do not conjoin (as ¬ a contradicts a)
¬ a	b	and	¬ a	¬ b	do not conjoin (as b contradicts ¬ b)

Because no conjunction yields a model, the overall model of the premises is therefore:

nil

The mental models yield the model, *a*, but the fully explicit models show that this model is erroneous, and that the premises contradict one another, yielding the null model.

The negation of a set of models takes into account default possibilities. Consider, first, the affirmation of an exclusive disjunction:

C2. Either A or else B.

It refers to a conjunction of two default possibilities, which have the fully explicit models:

a	¬ b
¬ a	b

If further knowledge shows that *A* is impossible, it eliminates the first of these possibilities, but it does not refute the conjunction of possibilities, because each possibility holds in default of knowledge to the contrary. Only if knowledge refutes both possibilities is the disjunction false, and so its models call for negation. The *partition* for one or more assertions is the set of all possible cases based on the affirmation and negation of the simple clauses in the assertions. So, the partition for (C2) has the set of models:

a	b
a	¬ b
¬ a	b
¬ a	¬ b

The negation of (C2) eliminates its models above from the partition, and yields the remainder:

a	b
¬ a	¬ b

In short, the negation of (C2) yields the complement of its set of models. An assertion that is a self-contradiction, such as:

C3. Both A and not-A

yields the null model: nil. And a premise that denies a self-contradiction:

C4. Not both A and not-A

is a tautology, which is bound to be true, and so the negation of *nil* is *T*, which stands for a tautology, and the negation of *T* is *nil*.

The mechanism for modulation, which is implemented in mSentential (see Khemlani et al., 2018), uses fully explicit models in knowledge to alter the interpretation of sentential connectives. It can block the construction of models of possibilities and it can add relations between their elements.

Consider the assertion:

C5. If it's raining then it's pouring.

Without modulation, its interpretation allows for the possibility that it is not raining and pouring. But, you know that *pouring* implies *raining*. This knowledge can be represented in fully explicit models as:

pouring	raining
¬ pouring	raining
¬ pouring	¬ raining

The conjunction of these models with the three models for C5 yields a biconditional interpretation:

raining	pouring
¬ raining	¬ pouring

which rules out the possibility that it is pouring but not raining.

References

Adams, E. W. (1998). *A primer of probability logic.* Stanford, CA: Center for the Study of Language and Information.

Austin, J. L. (1975). *How to do things with words.* New York: Oxford University Press.

Baratgin, J., Douven, I., Evans, J. St. B. T., Oaksford, M., Over, D., Politzer, G., & Thompson, V. (2015). The new paradigm and mental models. *Trends in Cognitive Sciences, 19*, 547–548.

Barnes, J. (Ed.). (1984). *The complete works of Aristotle*. Princeton, NJ: Princeton University Press.

Barrouillet, P., Grosset, N., & Lecas, J. F. (2000). Conditional reasoning by mental models: Chronometric and developmental evidence. *Cognition, 75*, 237–266.

Bear, A., & Knobe, J. (2017). Normality: Part descriptive, part prescriptive. *Cognition, 167*, 25–37.

Bell, V., & Johnson-Laird, P. N. (1998). A model theory of modal reasoning. *Cognitive Science, 22*, 25–51.

Bucciarelli, M., & Johnson-Laird, P. N. (2005). Naive deontics: a theory of meaning, representation, and reasoning. *Cognitive Psychology, 50*, 159–193.

Bucciarelli, M., Khemlani, S., & Johnson-Laird, P. N. (2008). The psychology of moral reasoning. *Judgment and Decision Making, 3*, 121–139.

Byrne, R. M. J. (2005). *The rational imagination: How people create alternatives to reality.* Cambridge, MA: MIT.

Byrne, R. M. J., Goodwin, G. P., Johnson-Laird, P. N., Khemlani, S. S., Quelhas, A. C., & Ragni, M. (2019). The real meaning of 'If'. (Under submission).

Byrne, R. M. J., & Johnson-Laird, P. N. (2019). If and or: real and counterfactual possibilities in their truth and probability. (Under submission).

Byrnes, J. P., & Beilin, H. (1991). The cognitive basis of uncertainty. *Human Development, 34*, 189–203.

Cariani, F. (2017). Choice points for a modal theory of disjunction. *Topoi, 36*, 171–181.

Cesana-Arlotti, N., Martín, A., Téglás, E., Vorobyova, L., Cetnarski, R., & Bonatti, L. L. (2018). Precursors of logical reasoning in preverbal human infants. *Science, 359*, 1263–1266.

Cheng, P. W., & Holyoak, K. J. (1985). Pragmatic reasoning schemas. *Cognitive Psychology, 17*, 391–416.

Conan Doyle, A. (1981). *The complete Sherlock Holmes.* Harmondsworth, Middlesex: Penguin.

Cruz, N., Over, D. E., & Oaksford, M. (2017). The elusive oddness of or-introduction. In G. Gunzelmann, A. Howes, T. Tenbrink, & E. J. Davelaar (Eds.). *Proceedings of the 32nd annual conference of the cognitive science society* (pp. 259–264). Austin, TX: Cognitive Science Society.

Evans, J. S. B. T. (2008). Dual-processing accounts of reasoning, judgment and social cognition. *Annual Review of Psychology, 59*, 255–278.

Evans, J. St. B. T. (2012). Questions and challenges for the new psychology of reasoning. *Thinking & Reasoning, 18*, 5–31.

Evans, J. St. B. T., Handley, S. J., Harper, C. N. J., & Johnson-Laird, P. N. (1999). Reasoning about necessity and possibility: A test of the mental model theory of deduction. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 25*, 1495–1513.

Frege, G. (1967). Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought. (Trans. Bauer-Mengelberg, S.). In J. Van Heijenoort (Ed.). *From Frege to Gödel: A source book in mathematical logic, 1879-1931* (pp. 1–82). Cambridge, MA: Harvard University Press (Originally published in 1879.).

Frosch, C. A., & Johnson-Laird, P. N. (2011). Is everyday causation deterministic or probabilistic? *Acta Psychologica, 137*, 280–291.

Fugard, A. J. B., Pfeifer, N., Mayerhofer, B., & Kleiter, G. (2011). How people interpret conditionals: Shifts toward the conditional event. *Journal of Experimental Psychology: Learning, Memory, & Cognition, 37*, 635–648.

Geurts, B. (2005). Entertaining alternatives: Disjunctions as modals. *Natural language semantics, 13*, 383–410.

Girle, R. (2009). *Modal logics and philosophy* (2nd ed.). London: Routledge.

Goldvarg, Y., & Johnson-Laird, P. N. (2000). Illusions in modal reasoning. *Memory & Cognition, 28*, 282–294.

Goldvarg, Y., & Johnson-Laird, P. N. (2001). Naive causality: A mental model theory of causal meaning and reasoning. *Cognitive Science, 25*, 565–610.

Goodwin, G. P., & Johnson-Laird, P. N. (2018). The truth of conditional assertions. *Cognitive Science, 42*, 2502–2533.

Grice, H. P. (1989). *Studies in the way of words.* Cambridge, MA: Harvard University Press.

Hinterecker, T., Knauff, M., & Johnson-Laird, P. N. (2016). Modality, probability, and mental models. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 42*, 1606–1620.

Hinterecker, T., Knauff, M., & Johnson-Laird, P. N. (2018). How to infer possibilities: A reply to Oaksford et al. (2018). *Journal of Experimental Psychology: Learning, Memory, and Cognition, 42*, 298–301.

Hintikka, J. (1962). *Knowledge and belief.* Ithaca, NY: Cornell University Press.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence.* London: Routledge & Kegan Paul.

Isard, S. D. (1975). What would you have done if? *Theoretical Linguistics, 1*, 233–255.

Jeffrey, R. (1981). *Formal logic: Its scope and limits* (2nd ed.). New York: McGraw-Hill.

Johnson-Laird, P. N. (1978). The meaning of modality. *Cognitive Science, 2*, 17–26.

Johnson-Laird, P. N. (1983). *Mental models.* Cambridge: Cambridge University Press.

Johnson-Laird, P. N. (2006). *How we reason.* New York: Oxford University Press.

Johnson-Laird, P. N., & Byrne, R. M. J. (1991). *Deduction.* Hillsdale, N.J.: Erlbaum.

Johnson-Laird, P. N., & Byrne, R. M. J. (2002). Conditionals: A theory of meaning, pragmatics, and inference. *Psychological Review, 109*, 646–678.

Johnson-Laird, P. N., Girotto, V., & Legrenzi, P. (2004). Reasoning from inconsistency to consistency. *Psychological Review, 111*, 640–661.

Johnson-Laird, P. N., & Hasson, U. (2003). Counterexamples in sentential reasoning. *Memory & Cognition, 31*, 1105–1113.

Johnson-Laird, P. N., Legrenzi, P., Girotto, V., Legrenzi, M., & Caverni, J.-P. (1999). Naive probability: A mental model theory of extensional reasoning. *Psychological Review, 106*, 62–88.

Johnson-Laird, P. N., Lotstein, M., & Byrne, R. M. J. (2012). The consistency of disjunctive assertions. *Memory & Cognition, 40*, 769–778.

Johnson-Laird, P. N., & Savary, F. (1999). Illusory inferences: A novel class of erroneous deductions. *Cognition, 71*, 191–229.

Johnson-Laird, P. N., & Tagart, J. (1969). How implication is understood. *American Journal of Psychology, 82*, 367–373.

Johnson-Laird, P. N., & Wason, P. C. (1970). A theoretical analysis of insight into a reasoning task. *Cognitive Psychology, 1*, 134–148.

Juanich, M., Teigen, K. H., & Gourdon, A. (2013). Top scores are possible, bottom scores are certain (and middle scores are not worth mentioning): A pragmatic view of verbal probabilities. *Judgment and Decision Making, 8*, 345–364.

Kahneman, D. (2011). *Thinking fast and slow.* New York: Farrar, Strauss, Giroux.

Karttunen, L. (1972). Possible and must. In J. P. Kimball (Ed.). *Syntax and semantics* (pp. 1–20). New York: Seminar Press.

Keynes, J. M. (1921). *A treatise on probability.* London: Macmillan.

Khemlani, S., Barbey, A. K., & Johnson-Laird, P. N. (2014). Causal reasoning: mental computations, and brain mechanisms. *Frontiers in Human Neuroscience, 8*, 1–15. https://doi.org/10.3389/fnhum.2014.00849.

Khemlani, S., Byrne, R. M. J., & Johnson-Laird, P. N. (2018). Facts and possibilities: A model-based theory of sentential reasoning. *Cognitive Science, 1–38*. https://doi.org/10.1111/cogs.12634.

Khemlani, S., Goodwin, G. P., & Johnson-Laird, P. N. (2015). Causal relations from kinematic simulations. In R. Dale, C. Jennings, P. Maglio, T. Matlock, D. Noelle, A. Warlaumont, & J. Yoshimi (Eds.). *Proceedings of the 37th annual conference of the cognitive science society*. Austin, TX: Cognitive Science Society.

Khemlani, S., Hinterecker, T., & Johnson-Laird, P. N. (2017). The provenance of modal inference. In Gunzelmann, G., Howes, A., Tenbrink, T., and Davelaar, E. J., (Eds.), Proceedings of the 32nd annual conference of the cognitive science society, pp. 663–668.

Khemlani, S. S., & Johnson-Laird, P. N. (2009). Disjunctive illusory inferences and how to eliminate them. *Memory & Cognition, 37*, 615–623.

Khemlani, S., & Johnson-Laird, P. N. (2012). Hidden conflicts: Explanations make inconsistencies harder to detect. *Acta Psychologica, 139*, 486–491.

Khemlani, S. S., & Johnson-Laird, P. N. (2013). The processes of inference. *Argument and*

Computation, 4, 4–20.
Khemlani, S. S., & Johnson-Laird, P. N. (2017). Illusions in reasoning. *Minds and Machines, 27*, 11–35.
Khemlani, S. S., Lotstein, M., & Johnson-Laird, P. N. (2012). The probability of unique events. *PLOS-ONE, 7*, 1–9.
Khemlani, S. S., Lotstein, M., & Johnson-Laird, P. N. (2015). Naive probability: Model-based estimates of unique events. *Cognitive Science, 39*, 1216–1258.
Khemlani, S., Lotstein, M., Trafton, J. G., & Johnson-Laird, P. N. (2015). Immediate inferences from quantified assertions. *Quarterly Journal of Experimental Psychology, 68*, 2073–2096. https://doi.org/10.1080/17470218.2015.1007151.
Khemlani, S. S., Mackiewicz, R., Bucciarelli, M., & Johnson-Laird, P. N. (2013). Kinematic mental simulations in abduction and deduction. *Proceedings of the National Academy of Sciences, 110*, 16766–16771.
Khemlani, S. S., Orenes, I., & Johnson-Laird, P. N. (2014). The negations of conjunctions, conditionals, and disjunctions. *Acta Psychologica, 151*, 1–7.
Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of measurement.* Academic Press.
Kratzer, A. (1977). What 'must' and 'can' must and can mean. *Linguistics and Philosophy, 1*, 337–355.
Kratzer, A. (1981). The notional category of modality. In H. J. Eikmeyer, & H. Rieser (Eds.). *Words, worlds, and contexts: New approaches in word semantics* (pp. 38–74). Berlin: de Gruyter.
Kratzer, A. (1991). Modality. In A. von Stechow, & D. Wunderlich (Eds.). *Semantics: An International Handbook of Contemporary Research* (pp. 639–650). New York: Mouton de Gruyter.
Kratzer, A. (2012). The notional category of modality. In A. Kratzer (Ed.). *Modals and conditionals: New and revised perspectives* (pp. 21–69). Oxford: Oxford University Press.
Kripke, S. (1963). Semantical considerations on modal logic. *Acta Philosophica Fennica, 16*, 83–94.
Lassiter, D. (2017). *Graded modality: Qualitative and quantitative perspectives.* New York: Oxford University Press.
Lewis, C. I., & Langford, C. H. (1932). *Symbolic logic.* New York: Century.
Lewis, D. (1981). Ordering semantics and premise semantics for counterfactuals. *Journal of Philosophical Logic, 10*, 217–234.
Lewis, D. (2000). Semantic analyses for dyadic deontic logic. In D. Lewis (Ed.). *Papers in ethics and social philosophy* (pp. 5–19). Cambridge: Cambridge University Press (Originally published 1974.).
McDermott, D. (1982). Nonmonotonic logic II: Nonmonotonic modal theories. *Journal of the Association of Computing Machinery, 29*, 33–57.
Meyer, J.-J. Ch., & van der Hoek, W. (1995). *Epistemic logic for AI and computer science.* Cambridge: Cambridge University Press.
Mill, J. S. (1874). *A system of logic* (8th ed.). New York: Harper.
Miller, G. A., & Johnson-Laird, P. N. (1976). *Language and perception.* Cambridge: Cambridge University Press. Cambridge, Mass.: Harvard University Press.
Moens, M., & Steedman, M. (1988). Temporal ontology in natural language. *Computational Linguistics, 2*, 15–21.
Mosteller, F., & Youtz, C. (1990). Quantifying probabilistic expressions. *Statistical Sciences, 5*, 2–34.
Oaksford, M., & Chater, N. (2007). *Bayesian rationality.* Oxford: Oxford University Press.
Orenes, I., & Johnson-Laird, P. N. (2012). Logic, models, and paradoxical inferences. *Mind & Language, 27*, 357–377.
Osherson, D. N. (1975). *Logical abilities in children, Vol. 3: Reasoning in adolescence.* Hillsdale, NJ: Lawrence Erlbaum Associates.
Osherson, D. N. (1976). *Logical abilities in children, Vol. 4: Reasoning and concepts.* Hillsdale, NJ: Lawrence Erlbaum Associates.
Over, D. E. (2009). New paradigm psychology of reasoning. *Thinking & Reasoning, 15*, 431–438.
Palmer, F. R. (2001). *Mood and modality* (2nd ed.). Cambridge: Cambridge University Press.
Partee, B. H. (1979). Semantics – mathematics or psychology? In R. Bäuerle, U. Egli, & A. von Stechow (Eds.). *Semantics from different points of view.* Berlin: Springer-Verlag.
Pearl, J. (2009). *Causality: Models, reasoning, and inference* (2nd ed.). New York: Cambridge University Press.
Peirce, C. S. (1931-1958). Collected papers of Charles Sanders Peirce. 8 vols. In Hartshorne, C., Weiss, P., & Burks, A. (Eds.), Cambridge, MA: Harvard University Press.
Phillips, J., & Knobe, J. (2018). The psychological representation of modality. *Mind and Language, 33*, 65–94. https://doi.org/10.1111/mila.12165.
Piaget, J. (1957). *Logic and psychology.* Oxford, UK: Basic Books.
Piéraut-Le Bonniec, G. (1980). *The development of modal reasoning: Genesis of necessity and possibility notions.* New York: Academic Press.
Portner, P. (2009). *Modality.* New York: Oxford University Press.
Quelhas, A. C., Rasga, C., & Johnson-Laird, P. N. (2017). A Priori true and false conditionals. *Cognitive Science, 41*, 1003–1030.
Ragni, M., & Johnson-Laird, P. N. (2018). Reasoning about possibilities: human reasoning violates all normal modal logics. *Proceedings of the 33rd annual conference of the cognitive science society* (pp. 2309–2314). Austin, TX: Cognitive Science Society.
Ragni, M., & Johnson-Laird, P. N. (2019). Reasoning about possibilities. (In preparation).
Ragni, M., Kola, I., & Johnson-Laird, P. N. (2018). On selecting evidence to test hypotheses. *Psychological Bulletin, 144*, 779–796.
Redshaw, J., Suddendorf, T., Neldner, K., Wilks, M., Tomaselli, K., Mushin, I., & Nielsen, M. (2019). Young children from three diverse cultures spontaneously and consistently prepare for alternative future possibilities. *Child Development, 90*, 51–61.
Rips, L. (1994). *The psychology of proof.* Cambridge, MA: MIT Press.
Rullmann, H., Matthewson, L., & Davis, H. (2008). Modals as distributive indefinites. *Natural Language Semantics, 16*, 317–357.
Shtulman, A. (2009). The development of possibility judgment within and across domains. *Cognitive Development, 24*, 293–309.
Sowa, J. F. (2018). Reasoning with diagrams and images. *Journal of Applied Logic* in press.
Stanovich, K. E. (1999). *Who is rational? Studies of individual differences in reasoning.* Mahwah, NJ: Erlbaum.
Steedman, M. (1977). Verbs, time and modality. *Cognitive Science, 1*, 216–234.
Stevens, S. S. (1946). On the theory of scales of measurement. *Science, 103*(2684), 677–680.
Tarski, A. (1956). The concept of truth in formalized languages. In A. In Tarski (Ed.). *Logic, semantics, metamathematics: Papers from 1923 to 1938. (Originally published in 1936.).* Oxford: Oxford University Press.
van Fraasen, B. (1973). The logic of conditional obligation. *Journal of Philosophical Logic, 1*, 417–438.
von Fintel, K., & Gillies, A. S. (2010). Must...stay...strong!. *Natural Language Semantics, 18*, 351–383. https://doi.org/10.1007/s11050-010-9058-2.
von Mises, R. (1928). *Wahrscheinlichkeit, Statistik, und Wahrheit.* Vienna: Springer.
von Wright, G. H. (1969). *An essay on deontic logic and the theory of action.* Amsterdam: North-Holland 1969.
Wertheimer, R. (1972). *The significance of sense: Meaning, modality and morality.* Ithaca, NY: Cornell University Press.
White, A. R. (1975). *Modal thinking.* Ithaca, NY: Cornell University Press.
Zimmermann, T. E. (2000). Free choice disjunction and epistemic possibility. *Natural Language Semantics, 8*, 255–290.