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The mental model theory of free choice permissions and paradoxical disjunctive inferences

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ABSTRACT

Inferences of the sort: *A or B*; therefore *A*, are invalid. Yet, the paradoxes of free choice are acceptable: *You can have sole or lobster*; so, *you can have sole*. Pragmatic theories attempt to save logic. A semantic theory of human reasoning is founded on mental models of possibilities. "Or" refers to a conjunction of possibilities that each hold in default of knowledge to the contrary. A disjunction: *it is permissible to do A or to do B,* yields a deontic interpretation of the possibilities, and elicits mental models of a conjunction of default permissions. They yield *or*-deletions, such as: therefore, *it's permissible to do A*. The theory predicts the paradoxes and new phenomena, which four experiments corroborated. For example, exclusive disjunctions such as: *Few of the artists are brutalists or else cubists*, have an intuitive model that yields *or*-deletions, but deliberation can construct models that refute them.

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The waiter offers you a choice of main course:

> You can have sole or lobster.
> You infer: I can have sole.

Everyone makes inferences of this sort, but as logicians pointed out they are invalid in orthodox modal logics for permissions (Ross, 1941; von Wright, 1969)—they are paradoxes of "free choice permission". Our term for them is more general, "*or*-deletion", because the paradoxes occur for topics other than permissions, e.g.:

> It may snow or it may rain.
> ∴ It may snow.

where "∴" stands for "therefore". We define an *or*-deletion as an inference from a premise containing "or", to a conclusion of one of the disjuncts in the premise, or their conjunction.

Inferences are *valid* in orthodox logics if their conclusion holds in all cases in which their premises hold (Jeffrey, 1981, p. 1), and the same idea holds for everyday reasoning except that the premises should be consistent—a constraint that logic does not require. Orthodox logics recognise two meanings for disjunctions. Inclusive disjunctions: *A or B or both,* are true when one or both their disjuncts, *A* and *B*, are true, otherwise, they are false; exclusive disjunctions: *A or else B, but not both*, are true when one and only one of their disjuncts is true, otherwise, they are false. This semantics is "truth-functional", because whether a disjunction is true or false is a function of the truth values of its two disjuncts. In orthodox modal logics (e.g. Hughes & Cresswell, 1996), which deal with possibilities of various sorts, *or*-deletions are therefore invalid. For instance, the premise for the inference above: *It may snow or it may rain* could be true even if snow is impossible. All it needs is for rain to be possible. Given these conditions, the disjunctive premise is true, but the conclusion that it may snow is false. A true premise and a false conclusion is a sure sign of invalidity.

Another way to express a truth-functional "or" is in a metadisjunction. An orthodox logic has a metalanguage including the predicates "true" and "false"

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in which to state its semantics (Tarski, 1944). English incudes these terms, and so an exclusive "or" can occur in a metadisjunction, such as:

One of the following assertions is true, and one of them is false:

It may snow.

It may rain.

As we show, metadisjunctions yield *or*-deletions that pragmatic theories cannot explain.

A corollary of truth-functional semantics is that inferences that are the converse of *or*-deletions, and that are known as *or*-introductions, are valid:

It may snow.

∴ It may snow or it may rain, or both.

Given that the premise is true, it establishes the truth of the first disjunct in the conclusion, which suffices for the disjunction to be true. Yet, *or*-introductions can be paradoxical too (Ross, 1941):

It is obligatory for you to tip the waiter.

∴ It is obligatory you to tip the waiter or to leave without paying, or both.

In general, any logic based on the truth-functional semantics for disjunctions implies that *or*-deletion is invalid and *or*-introduction is valid; but the pattern of inferences in daily life seems the exact opposite. That is the puzzle at stake. And, ever since Kamp (1973) revived interest in it, attempts at its solution have created a huge theoretical literature. It is far too large to digest here, but theories reflect two principal views. One maintains the truth-functional semantics of orthodox logics and aims to explain *or*-deletions on pragmatic grounds; the other abandons truth-functional semantics in favour of one in which certain *or*-deletions are valid. Our theory is of this sort, and it bases reasoning on mental models. But, it has an emergent property: it explains *or*-deletions and makes novel predictions about them. So, after a review of previous studies, the paper presents the mental model theory. It reports four corroboratory experiments. Finally, it shows their results are contrary to pragmatics theories.

Previous theories of *or*-deletions

Two principal sorts of theory, pragmatic and semantic, offer explanations of *or*-deletions (see Aloni, 2016; and Starr, 2016; for reviews). It is impossible to consider them all, and so this section outlines representative examples that our experiments address.

Pragmatic theories of or*-deletion*

Grice (1989) maintains that disjunctions and other connectives in natural language have the truth-functional meanings of logic, and that arguments to the contrary neglect the conventions of cooperative discourse (Grice, 1989, p. 24). Assertions can convey more than their literal meanings, and they do so when these conventions yield "conversational implicatures" (p. 26). One convention is that speakers should not assert less than they know. So, when a speaker asserts:

My wife is either in Oxford or Cambridge

he implicates that he does not know which of the two cities his wife is in (p. 8). Otherwise, he would have named a single city. Conversational implicatures can be canceled (p. 44), as when the speaker adds a coda:

And I know which.

Cancellability, however, is not a decisive test for implicatures (p. 44). Yet, they are defeasible (or "nonmonotonic") in that they can be withdrawn without contradiction. Indeed, a cogent view about reasoning in daily life is that it is always nonmonotonic, and so, given knowledge to the contrary, even valid conclusions can be withdrawn (e.g. Johnson-Laird et al., 2004; Marek & Truszynski, 2013; McDermott & Doyle, 1980). In contrast, orthodox logics are monotonic, and a contradiction validly implies any conclusion whatsoever.

Subsequent theories have developed Gricean pragmatics in myriad ways. They often rely on a "possible worlds" semantics. It postulates that an assertion: *It is possible that A*, is true in the real world if *A* itself is true in at least one of the possible worlds that determine the truth or falsity of assertions in the real world. Likewise, *It is necessary that A*, is true if *A* itself is true in all such possible worlds. The worlds that determine truth values for a given world are referred to as "accessible" from that world, and the nature of this relation of accessibility corresponds to different sorts of modal logic (Kripke, 1963). Because each accessible world has to determine the truth or falsity of any assertion about the given world, they are—as Partee (1979)

remarked—too big to fit inside anyone's head. So, any good theory of human understanding needs a plausible alternative to possible worlds.

Cognitive scientists have used Gricean principles as a basis for their pragmatics (e.g. Grodner et al., 2010; Levinson, 2000; Sperber & Wilson, 1995). Formal semanticists have made them more precise (e.g. Gazdar, 1979; Sauerland, 2004; Schulz, 2005). And one sensible strategy is to use an implicature to provide a missing premise, which then allows valid inferences in normal modal logics. Thus, Kratzer and Shimoyama (2002) argued that when the waiter offers you a choice of sole or lobster, he implicates that one choice is possible if and only if the other choice is possible. The addition of this premise to the waiter's remark yields:

> You can have sole if and only if you can have lobster. [$\Diamond A \leftrightarrow \Diamond B$]
>
> You can have sole or you can have lobster, or both. [$\Diamond A \vee \Diamond B$]
>
> ∴ You can have sole and you can have lobster. [$\therefore \Diamond A \wedge \Diamond B$]

(Sentences in brackets, which we use from time to time, are in modal logic: $\Diamond$ for *possible*, $\leftrightarrow$ for *if and only if*, $\vee$ for inclusive *or*, and $\wedge$ for *and*. Readers averse to symbols can ignore them.) The inference is valid in all normal modal logics, even its simplest, system K, which makes no assumptions about accessibility.

Various phenomena are consistent with a pragmatic account. One is that implicatures do not occur for obligations, so the following sort of *or*-deletion is invalid:

> You must borrow money or declare bankruptcy.
> ∴ You must declare bankruptcy.

Another is that denials of disjunctions do not evoke implicatures (Alonso-Ovalle, 2008). Hence, a denial such as:

> You can't have sole or lobster

can imply a conjunction of negations: You can't have sole and you can't have lobster.

One difficulty for Gricean theory is that utterances as a whole, not their constituents, elicit conversational implicatures (Cohen, 1971). Another difficulty, as a highly original post-Gricean theory emphasized (Fox, 2007), is that implicatures can arise within assertions, e.g.:

> Everybody knows that you can have sole or lobster.
>
> ∴ Everybody knows that you can have sole.

To cope with such *or*-deletions, these theorists introduced an exhaustivity operator akin to "only", and so:

> John talked to Mary or Sue.

has the implicature:

> John only talked to Mary OR Sue.

If, for all the alternatives to "or" in the implicature, which include "and", the resulting sentence is true, then it follows from the original disjunction (Fox, 2007, example 12). The operator can apply to constituents of sentences, such as the *that*-complement in:

> Everybody knows that you can have sole or lobster.

The exhaustivity operator is a covert lexical item that the grammar introduces to modify constituents, and a logical consequence of such grammatical structures are *or*-deletions (Chierchia et al., 2012). The post-Gricean theory predicts that these *or*-deletions occur in two contexts. One context is a disjunction in the scope of a plural quantifier based on "some" (Fox, 2007, p. 15), e.g.:

> Some students delayed the project or never finished it.
>
> ∴ Some students delayed the project.

The other context is the modal assertion of a possibility, because its truth depends on at least *some* possible world. *Or*-deletions occur for disjunctions with a narrow scope within the predicate of a modal assertion:

> It is possible that A or B,

but not for wide-scope disjunctions between two modal clauses:

> It is possible that A or it is possible that B.

Fox (2007) recognised that they do seem to occur in the latter case (Fox, 2007, p. 37), but did not offer any explanation, and his colleague confirmed this point: "We indeed don't predict free choice inferences with wide scope disjunction" (Moshe Bar-Lev, p.c., 12-26-2020). Its authors have modified the theory so that it no longer relies on multiple exhaustion operators (Bar-Lev & Fox, 2020, 2021). The local use of an operator remains distinct from Gricean implicatures from utterances as a whole. Other pragmatic theories have grown ever more complex, allowing implicatures to have implicatures

(Meyer, 2015) and game-theoretic interactions in discourse (Franke, 2011; van Rooij, 2010).

Implicatures and other additional premises can never convert a valid inference in orthodox logics into an invalid one. And so some other method is needed to explain why human reasoners tend to reject certain valid deductions in truth-functional logic, such as *or*-introductions. Grice's conversational maxims (van Rooij, 2010, p. 26) can be extended to inferences too: their conclusions should be informative, relevant, and truthful (Strawson, 1952, p. 91). So, *or*-introductions are not informative. The difficulty is that such a system may not be computable in a tractable way. That may be why theories of reasoning based on formal rules curb *or*-introductions (e.g. Rips, 1994).

Semantic theories of or*-deletion*

A variety of possible-world semantics exist to explain *or*-deletions (e.g. Aloni, 2007; Lewis, 1979; Simons, 2005; Veltman, 1996); other theories depend on dynamic logic (e.g. Roelofsen, 2013), or on alternative deontic logics (e.g. Dong & Roy, 2015; Ju & van Eijck, 2019). We focus, however, on a radical innovation—a new sort of meaning for disjunctions.

Zimmermann (2000, f.n. 6) argued that deontic and epistemic possibilities have a common semantic core. He postulated that both narrow-scope and wide-scope indicative disjunctions, such as:

> It is snowing or freezing.
>
> It is snowing or it is freezing.

refer to a conjunctive list of epistemic possibilities:

> It is possibly snowing, and possibly freezing, and possibly snowing and freezing.
>
> [◇snowing ∧ ◇freezing ∧ ◇(snowing ∧ freezing)]

When the clauses in disjunctions already contain modal terms, as in:

> It may be snowing or it may be freezing.

the conjunction contains adjacent pairs of modal operators in each of it conjuncts, one from "or" and one from the modal terms in the assertion:

> Possibly it's possibly snowing, possibly it's possibly freezing, and possibly it's possibly snowing and freezing. [◇◇snowing ∧ ◇◇freezing ∧ ◇◇(snowing ∧ freezing)]

These pairs can be reduced to single operators if individuals are self-reflective and know what they know, which can be formulated in an epistemic modal logic (Zimmermann, 2000, p. 284). The principle is irrevocable, and so epistemic *or*-deletions cannot be cancelled (Zimmermann, 2000, p. 284). For example:

He might be in Regent's Park or he might be in Victoria, and I know which

still yields the inference:

> ∴ He might be in Regent's Park.

This argument, if sound, refutes pragmatic approaches to epistemic *or*-deletions. The conjunctive lists could have their interpretation in normal modal logics or another interpretation, but the issue is open (Zimmermann, 2000, p. 263). Likewise, the lists of possibilities may be exhaustive or not (Zimmermann, 2000, p. 267). *Or*-deletion, however, does depend on the rejection of *or*-introduction as valid (Zimmermann, 2000, p. 274).

Zimmermann argues that deontic disjunctions depend on a principle akin to Kamp's (1979) appeal to the authority of the waiter to offer deontic choices. So, deontic *or*-deletions are valid only in certain circumstances, such as when a speaker uses a disjunction to create a permission, and not when a speaker only describes a deontic situation (Kamp, 1979, p. 255). Cancellation of deontic *or*-deletions, therefore, depends on speakers making clear their lack of authority by using such phrases as "but I forget which" (Kamp, 1979, p. 288).

Geurts (2005) agrees that disjunctions yield conjunctions of possibilities. He rejects Zimmermann's principles for reducing pairs of modals to single modals, and he emphasizes the common underlying meaning for modal operators. Overt pairs of modals, as in: *possibly it might be raining*, simply "fuse" into single modals in a conjunctive list (Geurts, 2005, p. 391). Fusion, we assume, does not occur for disjunctions concerning obligation, e.g.:

> It is obligatory for you to have sole or lobster.

But, the theory allows for *or*-deletions to have local effects (Geurts & van Tiel, 2013), as in:

> He knows that he can have sole or lobster.
>
> ∴ He knows that he can have sole.

Negation is a problem for treatments of disjunctions as lists of possibilities. As Geurts (2005, p. 406) writes: "The problem … is that a negated

disjunction should entail the negation of either disjunct. But since we are treating disjunctions as conjunctions it is unclear how we could capture this observation". Another problem is that if one of its conjuncts is false, then a conjunction itself – at least in a standard logic – is false too. So, a denial of one disjunct refutes the disjunction as a whole instead of yielding an inference.

Previous experiments on modal or*-deletions*

Empirical studies help to develop theories (Sauerland & Schumacher, 2016). Experiments have examined such matters as children's understanding of disjunctions as inclusive or exclusive (e.g. Noveck et al., 2002). But, no studies seem to have tested whether naive individuals accept inferences of *or*-deletions. Instead, they have asked participants to judge whether the first assertion "strongly suggested" the second assertion in such pairs as:

> John is allowed to give the teacher the dissertation or the commentary.
>
> John can choose which of the two he will give to the teacher.

The participants' ratings showed that they did accept such relations (Chemla, 2009).

Implicatures take time to process (e.g. Bott et al., 2012; Bott & Noveck, 2004; Breheny et al., 2006; Tomlinson et al., 2013). So, one question is whether *or*-deletions also take time. The first experimental study of deontic *or*-deletions examined children's judgments of the truth or falsity of disjunctions (Chemla & Bott, 2014). The children were told about a global evacuation from the earth, and that engineers were allowed to save man-made artifacts but not living creatures. They then judged whether the following assertion was true or false:

> Beverly-the-engineer is allowed to save a hammer or a lion.

For a truth-functional semantics, the correct response is "true", but *or*-deletion implies:

> Beverly-the-engineer is allowed to save a lion

which contravenes the deontic principle governing engineers, and so the children should respond "false". Most of them did so, and faster than they answered control questions. Subsequent experiments corroborated the occurrence of *or*-deletions and contrasted them with slower implicatures of other sorts. With limited time to respond, these other implicatures declined in frequency, but *or*-deletions did not. A study using the verification of assertions about pictures also showed that *or*-deletions were faster than other implicatures (van Tiel & Schaeken, 2017). None of these investigators, however, ruled out the following explanation. The construction of alternative utterances that the speaker might have asserted instead is negligible for *or*-deletions, because the two clauses in the disjunction make them explicit, but it can be costly for other sorts of implicature (see Pagliarini et al., 2018; Tieu et al., 2016, 2019). Indeed, the implicatures for *or*-deletions are stated in the disjunction itself (see Chierchia, 2006, 2013; Reinhart, 2006).

Chemla (2009) reports that *or*-deletions from quantified disjunctions, such as:

> Every customer can have sole or lobster. ∴ Every customer can have sole.

are just as reliable as their counterparts with singular subjects, such as "John". But, Geurts and Pouscoulous (2009) argue that this effect is a consequence of the speech act creating permissions, and so it should be reduced for other sorts of disjunction—a conjuncture that has received some corroboration (van Tiel, 2012).

Proponents of psychological theories based on probability rather than logic have argued that it is sensible for individuals to make an *or*-deletion from a disjunctive permission (Elqayam et al., 2010, p. 379). What yields these inferences, these authors claim, is pragmatics. To establish the probabilistic validity of the inference, pragmatics would have to ensure that the probability of the conclusion was no lower than the probability of the premise (Adams, 1998). But, these authors do not advocate any particular pragmatic theory. Moreover, speech acts referring only to probabilities cannot create permissions. These assertions have to refer to deontic possibilities. Possibilities are indeed more fundamental than probabilities. In summary, empirical studies have revealed the intricacies of testing implicatures, but they have yet to eliminate either pragmatic or semantic theories of *or*-deletions.

The mental model theory of *or*-deletions

The background

For many years, psychologists took for granted that human reasoning was based on orthodox logic (e.g. Braine & O'Brien, 1991; Inhelder & Piaget, 1958;

Rips, 1994). Experiments refuted this hypothesis. The difficulty of inferences did not depend either on the length of formal proofs designed to predict it, or on the number of cases that have to be considered in truth-functional semantics (for summaries, see Johnson-Laird & Byrne, 1991; Osherson, 1974). A contrasting theory is that the mind constructs mental models of the world, much as it does for perception, and draws conclusions from them. The early model theory was a humane way to represent truth-functional semantics (Johnson-Laird & Byrne, 1991; but cf. Byrne, 2005). We referred to it as the "model" theory, and we do here henceforth.

The new version of the model theory explains the role of possibilities in inferences depending on conditionals, disjunctions, and other compound assertions (Byrne & Johnson-Laird, 2019; Espino et al., 2020; Khemlani et al., 2018). It is implemented in a computer program, which yields *or*-deletions from modal assertions. The program, mModal, is at: https://github.com/CognitiveComputationLab/cogmods/tree/master/modal/student_projects/2019_guerth. Some critics see no point in such programs. But, if theories in cognitive science are not computable, their predictions may not follow from their principles. How do we know, say, that post-Gricean theory is computable? It probably is, but we can't be certain. And, no simple general test of computability can exist (see, e.g. Jeffrey, 1981, Sec. 6.4). So, a good way to show that a theory neither invokes processes that take too much for granted, nor omits something essential, is to implement it in a working computer program (Johnson-Laird, 1983, Ch. 1). The practical value of the program is that it allows users to check the theory's predictions. Otherwise, it is not easy to do so granted the multitude of inferences that people make; and the predictions sometimes surprise the programmers.

The model theory of possibilities

The new model theory bases reasoning on possibilities (Johnson-Laird et al., 2015). In almost any situation, real or imaginary, humans can envisage a small number of exhaustive and mutually exclusive finite alternatives, of which only one can occur (Johnson-Laird & Ragni, 2019). The toss of a coin, for instance, comes up heads or tails. Perhaps a bird could snatch the coin in mid-flight, and so people can invoke a "catch-all" category to include improbable events. Each alternative can occur in infinitely many ways—a coin has a particular size, weight, speed of rotation, trajectory, etc. Semanticists can conceive of all their different combinations as existing in possible worlds, but mental models cannot represent them. Instead, a finite model of an alternative represents only what is in common to all such realisations—to the point that people are happy to refer to it as a possibility, and to estimate its probability. These sets of alternatives are *primordial*, because they are the semantic basis of possibilities and probabilities. And they can have simple disjunctive descriptions, e.g.:

> The coin landed either heads or else tails.

Primordial situations are the basis of the underlying core sense of "possible". It can have at least three particular interpretations in daily life:

- *Alethic* possible and necessary consequences concern inferences or analyses, e.g.:

  > Viv is putting on her galoshes, and so it *may* follow that she is going out.

- *Deontic* possibilities and necessities concern permissible and obligatory actions or inactions, as in:

  > It is *permissible* for Viv to go out.

Certain speech acts can create permissions or obligations (Austin, 1975).

- *Epistemic* possibilities and necessities, or certainties, concern subjective probabilities in real or hypothetical situations (Lassiter, 2017), as in:

  > It is *very possible* that Viv went out.

A probability can be non-numerical as here, or numerical, e.g. "there's a probability of 75%".

Many cues in English help to create a particular interpretation of a core possibility. The use of a *that*-complement suggests an epistemic interpretation—*it is possible that she went out*, whereas the use of an *infinitival*-complement suggests a deontic interpretation—*it is possible for her to go out* (Johnson-Laird & Ragni, 2019). But, many assertions do not yield a particular interpretation. The police woman who asserts:

> You may cross the road now

may be reporting an inference, asserting a possibility, and giving you permission. So, "possible" is not ambiguous in the way in which, say, "duck"

is, but has a core meaning open to different interpretations (Johnson-Laird, 1978; Kratzer, 1977).

The meanings of disjunctions

The new model theory proposes a semantics for sentential connectives based on possibilities. In a development from its precursors (Battaglini, 2005; Geurts, 2005; Zimmermann, 2000), a primordial disjunction of indicative clauses:

It is hot or it is damp, or both

refers to an exhaustive conjunction of mutually exclusive possibilities and one impossibility, and each possibility holds in default of knowledge to the contrary. This concept of a default is familiar in theories of prototypical concepts, and it is built into object-oriented programming in which a class such as *birds* is treated as having *flying* as a default action. The class of *emus* overrules this default, but is still included in the class of birds (see, e.g. Russell & Norvig, 2003, p. 354). Given the disjunction that it is hot or damp or both, if you discover that it cannot be both, your knowledge eliminates the default possibility. Unlike logic, its falsity does not refute the conjunction, but just *modulates* the interpretation of the disjunction so it becomes exclusive. This feature of the theory avoids the problem of negation that beset Zimmermann's and Geurts's theories. For knowledge to refute an inclusive disjunction, it has to eliminate all three default possibilities, and thereby establish the one case that the disjunction ruled out. Because conjuncts referring to possibilities are defeasible, the semantics itself is nonmonotonic: it allows individuals to withdraw or to amend conclusions about possibilities (see, e.g. Marek & Truszynski, 2013). Indeed, reasoning in daily life is defeasible, and naive individuals withdraw even valid conclusions (Johnson-Laird et al., 2004; Khemlani & Johnson-Laird, 2011).

A major argument for implicatures is that speakers can cancel them. The model theory allows that knowledge can cancel default possibilities. In neither case is there a contradiction (cf. Skovgaard-Olsen et al., 2019): cancellation is neutral between pragmatic implicatures and default semantics. For default semantics, your knowledge has modulated the interpretation of an assertion. And modulation has been corroborated in many experiments (Johnson-Laird & Byrne, 2002), including the interpretations of disjunctions (Quelhas & Johnson-Laird, 2017), inserting a temporal and spatial relation between events (Juhos et al., 2012) and determining the truth or falsity of a disjunction solely from its meaning (Quelhas et al., 2019). One pertinent case concerns judgments of conditionals as true or false in virtue of their meanings (Quelhas et al., 2017; pace Quine, 1953), e.g.:

If Martha has flu then she is ill.

The model theory distinguishes between two systems of reasoning—an idea due to the late Peter Wason (e.g. Johnson-Laird & Wason, 1970), but that the model theory has always maintained (cf. Evans, 2008). System 1, the intuitive system, relies on models that represent only what is true in each possibility. Given a disjunction such as, *it is hot or damp or both*, the semantics for inclusive "or" yields such a model of a conjunction of three default possibilities:

hot	
	damp
hot	damp

Like the computer program implementing the theory, we use words instead of real models in this diagram, and we omit the signs for conjunction and for default possibilities. The model yields the following conclusions even though the disjunction itself makes no mention of possibilities:

∴ It is possible that it's hot.

∴ It is possible that it's damp.

∴ It is possible that it's hot and that it's damp.

People accept such conclusions (Hinterecker et al., 2016). They are so plausible that one critic suggested that they are valid in orthodox logic. In fact, they are invalid in all normal modal logics. A counterexample to the first of these inferences is that it is impossible for it to be hot but it is damp, and so the disjunctive premise is true in a truth-functional semantics, but the conclusion is false. Analogous counterexamples refute the other inferences. The conclusions also do not follow in probabilistic logic (Adams, 1998), and, as far as we can tell, only the model theory and its precursors predict them.

A deliberative process of reasoning, system 2, can construct explicit models that also represent an

exhaustive conjunction of default possibilities. In each possibility they represent what is true and also what is false, using true negations to do so. The explicit models for the preceding disjunction are, therefore:

hot	¬ damp
¬ hot	damp
hot	damp

where "¬" denotes "not", and its symbol is linked to a semantic procedure for negation. These models still yield the inferences of possibilities above. But, explicit models can correct certain fallacies that intuitive models lead to (see Khemlani & Johnson-Laird, 2017; and for an alternative account, Koralus & Mascarenhas, 2013).

The preceding theory (in Hinterecker et al., 2016; Johnson-Laird et al., 2015) needs only one further principle to predict *or*-deletions from modal assertions. The possibilities that "or" itself elicits are defaults. Explicit possibilities in a disjunction can replace them. We explain this point in the introductions to Experiments 1 and 2 that derive their predictions.

Quantifiers in the model theory

Quantifiers are phrases such as, "few of the artists", and the model theory predicts that reasoners will tend to infer *or*-deletions from certain quantified disjunctions rather than others. Before we reach these predictions in Experiment 4, we explain the theory's treatment of quantifiers (Khemlani & Johnson-Laird, 2021). The simplest logic for quantifiers is for those based on "some" and "all", whereas those based on "few", "most", and their cognates, call for a higher-order logic (Barwise & Cooper, 1981). These authors dealt with them as "generalized quantifiers", as Montague (1974) did for all English noun phrases, even proper nouns, such as, "Betty". Hence, some semanticists—including two reviewers of the present paper—suppose that this method is the only way to handle higher-order quantifiers. If that were true, it would be impossible for human beings to understand simple assertions, such as:

> Few of these artists are cubists.

As a generalised quantifier, "few of these artists", refers to a set of sets, i.e. those of which few of these artists are members. So, if the relevant artists are Dali, Ernst, Hockney, Kahlo, and Picasso, then the quantifier refers to the set of all sets to which few of these five belong, e.g. English painters, abstract artists, and … cubists. The predicate of a sentence refers to the relevant set, and so "cubists" refers to the set of cubists. And the sentence above is true because the set of cubists is in the set of sets of which few of these artists are members—only Picasso is a cubist. Even if you knew all the sets to which few of these five artists belong, it would take you far too long to call them to mind in order to understand "few of these artists". Generalised quantifiers, like possible worlds, which Montague also espoused, are a logical artifice in which uniform semantics takes precedence over psychological plausibility (Johnson-Laird, 1983, p. 180).

There is a different semantics for quantifiers—even for higher-order ones. A quantified sentence asserts a relation between the set to which the noun phrase in the quantifier refers and the set to which the predicate refers (Boole, 1854). So, "few of these artists are cubists" means that only a small subset of these artists are cubists. To follow Khemlani and Johnson-Laird (2021), it means:

> The number of these artists who are cubists ranges from zero to less that a quarter of the number of these artists.

$$[0 \leq |\text{ these artists} \cap \text{cubists }| < ¼ |\text{ these artists }|]$$

The value of zero allows for an interpretation equivalent to "few, if any", and the value of a quarter derives from empirical studies of the proportions that individuals assign to quantifiers (Moxey & Sanford, 1993).

Representations of this meaning and those for other quantifiers can be used to construct a model of a quantified assertion, and to check the accuracy of any modifications to the model. The details of the process of construction are described in Khemlani and Johnson-Laird (2021). So, here, we walk readers through the steps in the construction of intuitive models and then their deliberative counterparts.

Consider the assertion:

> At least some students chose acting.

It means that the number of students who chose acting ranges from two, for a plurality, to the number of students. The construction of an intuitive model (system 1) takes two steps. First, it models the

noun phrase in the quantifier, "students", and represents, say, three students:

student
student
student

Second, it picks out a subset that fits the meaning of the quantifier, and allocates the relation in the predicate, "chose acting", to them:

student	chose acting
student	chose acting
student	

This diagram denotes a single model of a situation in which each row represents an individual. Intuition copes only with a single model of a quantified assertion at a time (Khemlani & Johnson-Laird, 2021).

Deliberation (system 2), if it is evoked, seeks alternatives to intuitive models, often to refute or to refine conclusions drawn from them. Its input is usually an intuitive model and, where relevant, a putative conclusion. It can flesh out an intuitive model of an individual so that it is explicit. For example, given the preceding model, it turns it into an explicit model:

student	chose acting
student	chose acting
student	¬ chose acting

But, deliberation can also add new entities and properties to a model, modify their existing properties and relations—checking that each modification is faithful to the meaning of the relevant assertions. These processes are largely irrelevant to our present studies, but one aspect of disjunction is critical—its scope.

A negative disjunction, which is inclusive, such as:

None of the actors sings or dances

has two interpretations depending on scope. A narrow scope interpretation of "or" yields valid *or*-deletions in both the model theory and in logic, but a wide scope interpretation of "or" does not. Proofs of these two sorts of claims are in the supplemental materials (Part 1 of S0). If the disjunction is exclusive, not even the narrow scope disjunction yields valid *or*-deletions (see proof 1 in Part 2 of S0). The supplemental materials, which include the materials and results of the four experiments, are at https://osf.io/p7ucf/?view_only=e0d349a6770e4a56a3ce830296d2af8e.

Our four experiments tested the model theory's predictions about deontic disjunctions, epistemic disjunctions, conditionals embodying epistemic *or*-deletions, and quantified disjunctions. They each examined some predictions that diverge from normal modal logics and from pragmatic theories.

Experiment 1: *or*-deletions with deontic disjunctions

A narrow-scope deontic disjunction such as:

You're permitted to speak or to sing

elicits intuitive models of a conjunction of defaults of what you are permitted to do, which can include carrying out both actions:

speak	
	sing
speak	sing

A wide-scope deontic disjunction such as:

You're permitted to speak or you're permitted to sing

elicits the same models. It is also permissible for you not to carry out either action, which can be added to the explicit models of the assertion. No need exists to combine "permissible" with the "possible" that the disjunction elicits, or to fuse them together (pace Zimmermann, 2000; Geurts, 2005). Instead, as we mentioned earlier, the core possibilities that the disjunction elicits are defaults that in this case, deontic possibilities replace. The intuitive models, therefore, yield these *or*-deletions:

∴ You may speak.

∴ You may sing.

and even their conjunction.

Disjunctions about obligations yield only models of a conjunction of the three possibilities above. Hence:

You are obligated to speak or to sing

does not imply an *or*-deletion such as:

∴ You must speak.

As the models establish, you could meet the obligation by singing instead. What does follow is that you are obligated to carry out one of these actions. And it also follows that you are permitted to carry out each action, e.g.:

∴ You may speak.

The model theory therefore yields:

Prediction 1. Free choice permissions: Modal disjunctions are not primordial: no case is bound to

occur. Hence, individuals will accept *or*-deletions from disjunctive permissions, whether "or" has a narrow scope within a permission or a wide scope over two permissions.

The prediction is contrary to normal modal logics, and to post-Gricean theory (e.g. Fox, 2007; Bar-Lev & Fox, 2020, Sec. 2.3), which accepts deontic *or*-deletions only for narrow-scope disjunctions.

Previous empirical studies do not appear to have tested which immediate inferences of *or*-deletion occur from deontic premises. Experiment 1 made such a test. This experiment, like all our subsequent ones, was carried out in Portuguese, the native language of the participants, and so we have translated the materials here. Portuguese and English could differ in subtle ways so that the results do not generalise from one language to the other. However, in the experiment, a typical trial was:

Você tem obrigação de comprar ou pedir emprestado.

Conclui-se que você pode comprar?

We translate it as:

You are obligated to buy or to borrow.

Does it follow that you can buy?

No obvious difference in the sense of the Portuguese could account for results that would not occur for English. And the same judgment holds for the translations of "it is permissible that … ", and "it is possible that … " in this and later experiments (see the supplemental materials for all the translations).

Design and participants

The participants acted as their own controls and evaluated eight sorts of deontic *or*-deletions. Their premises referred to either permitted or obligatory actions, which were either narrow-scope or wide-scope disjunctions, and their conclusions referred to either permitted or obligatory actions (see Table 1 for a summary). Half of the conclusions that each participant evaluated referred to the first action in the disjunction and half of them referred to the second action. Two "filler" trials tested *or*-introductions, such as:

You are permitted to sing.

Table 1. The *or*-deletions from narrow-scope and wide-scope disjunctions of permissible and obligatory actions in Experiment 1 ($N = 173$), and the percentages of the model theory's predicted evaluations, where *A* and *B* stand for everyday actions, and "*A/B*" denotes that half the conclusions in the trials contained *A* and half of them contained *B*.

Deontic premises with narrow and wide scope disjunctions	Conclusions to be evaluated	Predicted evaluations and their percentages
You are permitted to do A or B.	*∴ You can do A/B.*	Yes: 98
	∴ You must do A/B	No: 96
You are permitted to do A or you are permitted to do B.	*∴ You can do A/B.*	Yes: 94
	∴ You must do A/B.	No: 94
You are obligated to do A or B.	*∴ You can do A/B.*	Yes: 78
	∴ You must do A/B.	No: 69
You are obligated to do A or you are obligated to do B.	*∴ You can do A/B.*	Yes: 72
	∴ You must do A/B.	No: 63

Note: The balances of percentages in each cell are for evaluations opposite to the predictions.

Does it follow that you are permitted to sing or to dance?

Their purpose was to inhibit the participants from always expecting the same sort of inference. They are valid in deontic logic, but not in the model theory, which allows that they follow only as possibilities, and so it predicts a "No" evaluation. The experiment tested four groups of participants of roughly equal numbers to counterbalance two factors. Each set of contents, and therefore each participant, had half the *or*-deletions with conclusions about the first action in the premise and half the *or*-deletions with conclusions about the second action in the premise. Two of the groups had one random assignment of these conclusions, and two of the groups had the complementary assignment of conclusions. Within each pair of these groups, one group had a random allocation of contents to the inferences, and one group had a complementary allocation. The ten inferences, which included the two filler trials, were presented in a different random order to each participant.

The experiment tested 173 psychology undergraduates (146 women and 27 men) from ISPA-IU, in Lisbon, who were volunteers. Their mean age was 20.1 years ($SD = 4.5$).

Materials and procedure

All the assertions in the inferences used the pronoun "you" as their subject to enhance a

deontic interpretation, and their infinitival verbs were simple intransitives, such as: "relax", "sing", and "drink" (the full set of contents in English and their original Portuguese are in S1 of the supplemental materials).

The participants were tested in small groups of about twenty individuals. They received a booklet with general instructions on the first page and then the ten inferences, each on a separate page. They had no time limit to complete the task. The key instructions were:

> You will encounter two sorts of conclusion: one sort is about whether given that an assertion is true, it follows that you may carry out a particular action. The other sort is about whether given that an assertion is true, it follows that you must carry out a certain action.

The participants responded either "Yes" or "No" by ticking an appropriate box, as here:

Imagine that someone told you:
You're permitted to speak or to sing.
Does it follow that you may speak? Yes ☐ No ☐

Results and discussion

The participants made more of the model theory's predicted evaluations than not: 161 participants out of 173 did so, four participants made more unpredicted evaluations than predicted ones, and the remaining eight participants were ties (Binomial test with a prior of .5, $p < 1$ in a million). The two assignments of clauses (*A* or *B*) to conclusions had no reliable effect on the percentages of predicted evaluations (Groups 1 and 2 = 78% versus Groups 3 and 4 = 79%; Mann Whitney Test, $z < .8$, $p > .43$, Cliff's $\delta < .06$). Likewise, the two different assignments of the contents to the inferences had no reliable effect on the percentages of predicted evaluations (Groups 1 and 3 = 77% versus Groups 2 and 4 = 80%; Mann Whitney test, $z < 1.4$, $p > .18$, Cliff's $\delta < .12$). We, therefore, amalgamated the results for further analysis. The two filler items of *or*-introductions tended to be rejected (60% overall), and the next experiment examined such inferences in detail.

Table 1 presents the percentages of predicted evaluations for the eight sorts of experimental inference: there was no reliable difference in the results within participants for conclusions about *A* and for conclusions about *B* (Wilcoxon test, $z = 1.028$, $p < .4$, Cliff's $\delta < .06$). The supplemental materials present the percentages of predicted evaluations for each inference in each of the four versions. As Table 1 suggests, a reliable difference in the percentages of predicted evaluations occurred between permissible actions (96%) and obligatory actions (71%; Wilcoxon test, $z = 8.34$, $p < .0001$, Cliff's $\delta = .45$). This difference has been observed in earlier studies of deontic reasoning (e.g. Bell & Johnson-Laird, 1998; Bucciarelli & Johnson-Laird, 2005), and may reflect the number of models that need to support an action. For instance, if you are permitted to sing, then this action can be only one of those that you are allowed to do. But, if you are obligated to sing, then this action must be one in all the actions that you are allowed to do. The difference in these results rules out a simple bias to respond "yes" to conclusions about what is permissible, and "no" to what is obligatory. As Table 1 shows, acceptances of permissibility were slightly greater for narrow-scope disjunctions (88%) than for wide-scope disjunctions (83%), and the difference was reliable, and for these inferences from premises concerning permissions (Wilcoxon tests, *z*'s > 2.0, *p*'s $< .04$ two tail, Cliff's $\delta > .10$), but not reliable for those inferences from premises concerning obligations (Wilcoxon test, $z = 1.4$, $p > .1$, Cliff's $\delta = .07$). We suspect that a disjunction of two modal clauses may have made the participants a little less certain in their evaluations, because they hint at two inconsistent alternatives. However, the general acceptance of *or*-deletions from wide-scope disjunctions bears out the model theory but is contrary to post-Gricean theory. Inferences that an action is permissible because it occurs in a disjunction of obligations also corroborate the model theory. These inferences from disjunctions are invalid in deontic logics (e.g. system D), and it is unclear whether pragmatic theories predict them.

Experiment 2: epistemic *or*-deletions and *or*-introductions

The aim of the experiment was to compare epistemic *or*-deletions of the sort:

> It is possible that A or B, or both.
>
> ∴ It may be that A.

with their converse *or*-introductions:

> It is possible that A.

∴ It may be that A or B, or both.

The experiment manipulated whether the disjunctions had a narrow or wide scope, and whether they were inclusive or exclusive. In the model theory, the inclusive disjunctions above have intuitive models of the sort:

A
B
A B

They yield *or*-deletions regardless of the scope of the disjunctions. Exclusive disjunctions do not have a model of the joint possibility of *A and B*, and so do not yield it as an *or*-deletion.

Consider this *or*-introduction:

> It is possible that the fire burnt down the yellow house.
>
> ∴ The fire may have burnt down the yellow house or the green house, or both.

Nothing in the premise implies the possibility that the fire burnt down the greenhouse. So, the model theory predicts a bias towards the rejection of *or*-introductions (Orenes & Johnson-Laird, 2012). The model theory, therefore, makes this prediction:

> Prediction 2. Epistemic *or*-deletions and *or*-introductions: *Or*-deletions for epistemic possibilities should occur for all sorts of disjunction, narrow or wide scope, and inclusive or exclusive; and *or*-deletions should be accepted more often than converse *or*-introductions.

The prediction contrasts with the status of these inferences in all normal modal logics: *or*-deletions are invalid, and *or*-introductions are valid for inclusive disjunctions but not for exclusive ones. Pragmatic theories can explain the rejection of *or*-introductions on the grounds that their conclusions are uninformative, but it is not clear how they could predict that they are more likely to be rejected than *or*-deletions.

Design, participants, materials, and procedure

The participants evaluated eight *or*-deletions, depending on whether the premise was an inclusive or exclusive disjunction, whether it was a narrow-scope or wide-scope disjunction, and whether the conclusion concerned a possibility ("may") or a necessity ("must"). Half of these conclusions referred to *A* and half of them referred to *B*. The participants evaluated four *or*-introductions, depending on whether the conclusion was an inclusive or exclusive disjunction, and on whether it was a narrow-scope or wide-scope disjunction. Half of these inferences had *A* as the premise and half of them had *B* as the premise. These *or*-introductions used only conclusions concerning a possibility ("may"). The 12 inferences were presented to each participant in a different random order.

Four different groups of participants carried out the experiment depending on two counterbalanced allocations of the contents to the inferences, and on two counterbalanced allocations of *A* and *B* from the disjunction to the single assertion in each inference, either the conclusion for *or*-deletions or the premise for *or*-introductions.

The experiment tested 139 new volunteers from the same population as before (115 women and 24 men; mean age 21.0 years, $SD = 6.2$).

We devised 12 sets of contents based on a grammatical subject of a transitive verb with two potential objects. The subjects referred either to humans, e.g. "the plumber", or to entities, e.g. "the fire", and the objects referred to entities (see Table S2 in the supplemental materials). The procedure and instructions were almost identical to those of Experiment 1 except for a slightly expanded account of the evaluation of the inferences:

> In carrying out these judgments, please be sure to take into account only what is explicitly asserted in the premises, and do not make plausible judgments based on your general knowledge. You should assume that the premise is true, and then, bearing that in mind, judge whether the conclusion is thereby guaranteed to be true too.

Results and discussion

The participants made more predicted evaluations (84%) than unpredicted evaluations (16%): 133 participants out of 139 did so, none made more unpredicted evaluations than predicted ones, and the remaining six participants were ties (Binomial test, $p < .5^{133}$). Neither the two assignments of contents nor the two allocations of *A* and *B* from the disjunction to the categorical assertion had a reliable effect on performance, and so we amalgamated the results for analysis (see the supplemental materials for the percentages of predicted evaluations for each inference in its four versions).

Table 2 presents the percentages of predicted evaluations for the 24 sorts of inference. Overall,

Table 2. The percentages of the model theory's predicted evaluations of epistemic *or*-deletions and or-*introductions* in Experiment 2 ($N = 139$), where *A* and *B* stand for everyday assertions.

Epistemic inferences	Premises	Conclusions	Percentages of predicted evaluations
Or-deletion	*Possible that A or B, or both.*	∴ *May A.*	Yes: 93 (89)
		∴ *May B.*	Yes: 91 (90)
		∴ *Must A.*	No: 94 (87)
		∴ *Must B.*	No: 86 (86)
Or-deletion	*Possible that A or B, but not both.*	∴ *May A.*	Yes: 97 (96)
		∴ *May B.*	Yes: 90 (86)
		∴ *Must A.*	No: 96 (93)
		∴ *Must B.*	No: 87 (96)
Or-introduction	*Possible that A.*	∴ *May A or B,*	No: 56 (54)
	Possible that B.	*or both.*	No: 77 (77)
Or-introduction	*Possible that A.*	∴ *May A or B,*	No: 80 (71)
	Possible that B.	*but not both.*	No: 77 (77)

Note: The balances of percentages in each cell are for evaluations opposite to the predictions.
The first percentages are for narrow-scope disjunctions: *Possible that A or B* and the second percentages (in parenthesis) are for wide-scope disjunctions: *Possible that A or possible that B.*

as the model theory predicts, participants accepted *or*-deletions (91%) significantly more often than *or*-introductions (29%; Wilcoxon test, $z = 6.676$, $p < .0001$, Cliff's $\delta > .61$). No reliable difference occurred in these percentages depending on whether the disjunctions had a narrow scope: *It is possible that A or B* (85%) or a wide-scope: *It is possible that A or it is possible that B* (83%; Wilcoxon test, $z < 1.45$, $p < .2$, Cliff's $\delta < .09$). There was a small but reliable difference in the predicted evaluations between exclusive disjunctions (87%) and inclusive disjunctions (82%; Wilcoxon test, $z = 3.12$, $p < .002$, Cliff's $\delta > .18$). The model theory has a long-standing explanation: exclusive disjunctions call for only two models of possibilities, whereas inclusive disjunctions call for three (Johnson-Laird et al., 1992). Post-Gricean theory (e.g. Bar-Lev & Fox, 2021) predicts the greater frequency of *or*-deletions from exclusive disjunctions, but it does not explain their robust occurrence from wide-scope inclusive disjunctions. Nor does this or other pragmatic accounts explain why people tend to reject *or*-introductions more often than *or*-deletions.

Experiment 3: the truth of conditionals embodying epistemic *or*-deletions

As we explained earlier, individuals judge certain conditionals to be true and others to be false on the basis of their meanings alone (Quelhas et al., 2017). So, they should evaluate those conditionals embodying epistemic *or*-deletions that are valid in the model theory as true too. For example, the following conditional embodies a valid *or*-deletion:

> If it is possible to buy a pair of sneakers or to buy a pair of sandals, then it is possible to buy a pair of sneakers.
>
> Is this assertion true or false?

The model theory therefore yields:

> Prediction 3. Truth of conditionals asserting valid *or*-deletions: Individuals should evaluate as true those conditionals embodying *or*-deletions that are valid in the model theory.

This prediction is contrary to normal modal logics. ($\Diamond(a \vee b) \rightarrow \Diamond a$ is not bound to be true: it has this counterexample: *a* is false in all accessible worlds, but *b* holds in one accessible world, and so the first clause is true, but the second clause is false.) The prediction is also contrary to Gricean theory, which cannot accommodate implicatures within assertions. But, post-Gricean theory makes the prediction (p.c., Danny Fox & Moshe Bar-Lev, 12/26/20). In the model theory, the impossibility of a disjunction of actions implies the impossibility of each of them, e.g.:

> If it is not possible to buy a pair of sneakers or to buy a pair of sandals then it is not possible to buy a pair of sneakers.

The truth of this conditional also holds in normal modal logics. ($\neg \Diamond(a \vee b) \rightarrow \neg \Diamond a$ is true: it is the contrapositive of *or*-introduction.)

These two preceding predictions concern *if*-clauses that are possible and *if*-clauses that are not possible. The experiment also examined *if*-clauses of a third sort: *possible not-A or not-B*. Given a *then*-clause of *possible A,* its presumption is that *possible not A* (see Ragni & Johnson-Laird, 2020), and so some participants will evaluate such conditionals as true. And a *then*-clause of *not possible A* is consistent

with *possible not A*, and so some participants will evaluate such conditionals as true too. Neither is an obvious valid inference from the *if*-clause, and so some participants may evaluate the conditionals as *false* or as *could be true or false*. Granted the long established difficulty of coping with negation (e.g. Wason & Johnson-Laird, 1972, Ch. 2), the theory, therefore, predicts a trend of decreasing evaluations of truth over the three sorts of *if*-clauses: affirmative, negative, and negated possibilities.

Design and participants

Participants acted as their own controls, and evaluated the truth values of 12 sorts of conditional based on three sorts of *if*-clause and four sorts of *then*-clause, where "*A/B*" denotes that half the clauses were with *A* and half of them were with *B*: Table 3 presents the 12 sorts of conditional used in the experiment.

Each participant carried out two evaluations of each of the 12 sorts of conditional with different contents, so that in the experiment as a whole each of four sets of contents occurred equally often in the twelve sorts of conditional. The participants received the problems in counterbalanced orders.

The experiment tested 70 undergraduates (56 women and 14 men; mean age was 22.5 years, *SD* = 8.6) from the same population as before. They received course credits for their participation.

Materials and procedure

The contents of the conditionals concerned everyday topics, such as shopping and traveling (see S3 in the supplemental materials). The experiment was carried out online using the Qualtrics site. The instructions stated that the participants had to judge each conditional as *true*, *false*, *could be true or false*, or *impossible to say*, as in this typical trial:

Imagine that someone told you:

If it is possible to buy a pair of sneakers or to buy a pair of sandals, then it is possible to buy

a pair of sandals.

Do you consider that this sentence is:

True ☐ False ☐ Could be true or false ☐
Impossible to decide ☐

The participants selected their option by moving the cursor to the relevant box and clicking.

Results and discussion

No reliable difference in percentages occurred between *then*-clauses referring to *A* and those referring to *B*, and so we amalgamated the results for analysis. Overall, 50 participants out of 70 made more predicted than non-predicted evaluations, 13 participants made more non-predicted than predicted evaluations, and 7 participants made tied evaluations (Binomial test, $p < .0001$). Table 3 presents the frequencies of the different evaluations for the 12 sorts of conditional. The conditionals affirming possibilities yielded more predicted evaluations (90%) than those negating possibilities (75%), which in turn yielded more predicted evaluations than those referring to the possibility of negated actions (33%). This trend held for 65 out of 70 participants, and only five participants failed to show it (Binomial test with a prior of .5, $p < 1$ in a million). The disjunctions of negative events led to the expected diversity of evaluations. Some followed the presumptions of the *then*-clauses, but a substantial majority of participants were split between either rejecting the conditional or judging that it could be true or false. The evaluation of conditionals instantiating simple affirmative epistemic *or*-deletions as true corroborates the model theory, but runs counter to

Table 3. The 12 sorts of epistemic conditionals embodying *or*-deletions in Experiment 3 ($N = 70$), and the percentages of judgments of them as "true", "false", and "could be true or false"; percentages in bold are for the model theory's predicted evaluations, and the symbol *A/B* in the *then*-clauses refers to *A* clauses on half the trials and *B* clauses on half the trials.

The conditionals to be judged true or false		Percentages of judgments		
If-clause	*then*-clause	True	False	Could be true or false.
If possible A or B	*then possible A/B.*	**94**	1	5
	then not possible A/B.	4	**85**	9
If not possible A or B	*then not possible A/B.*	**70**	10	11
	then possible A/B.	13	**75**	8
If possible not A or not B	*then not possible A/B.*	**45**	29	23
	then possible A/B.	**36**	39	20

Note: The balances of percentages in each row were evaluations of "impossible to decide", which participants chose on less than 10% of trials for each row in the table.

normal modal logics and to Gricean explanations. It is compatible, however, with post-Gricean theories.

Experiment 4: *or*-deletions from quantified assertions

This experiment examined *or*-deletions from six sorts of quantified exclusive disjunctions in which "or" has a narrow scope of the sort: *All of the A's did B or else C*. In logic, none of the six sorts yield valid *or*-deletions (see Part 2 of S0 in the supplemental materials for their counterexamples in logic, and Table 4 below for analogous counterexamples in the model theory). But, the model theory predicts that individuals will be more likely to accept *or*-deletions from those assertions based on *none*, *few*, and *some*, than from those based on *one*, *most*, and *all*. The difference cannot be a matter of logic. A simple way to understand it is to imagine a game of musical chairs in which *all the chairs are occupied by girls or else by boys*. If the girls occupy all the chairs then there is no room for the boys, and vice versa. The disjunction is primordial, and only one of its alternative possibilities can occur. But, suppose instead that *some of the chairs are occupied by girls or else by boys*. Now the chairs can accommodate both possibilities: the quantifier can refer to a small enough proportion for girls and for boys to sit on the chairs, i.e. a proportion of less than half. The disjunction is not primordial, and its intuitive model yields *or*-deletions.

The general principle for *or*-deletions is therefore that a single intuitive model has to represent all the individuals to which a quantifier refers together with their relations in the disjunctive predicate. For a predicate that is an exclusive disjunction, the crux is whether or not such a model can represent appropriate proportions of individuals with the two mutually exclusive predicates. The model can

Table 4. The premises of quantified exclusive disjunctions and their *or*-deletions in Experiment 4 ($N = 105$), examples of their single intuitive models and resulting evaluations, examples of their deliberative models and their consequences, and the percentages of predicted evaluations based on the intuitive models. *B/C* indicates that half the conclusions had *B* as predicate, and half had *C* as predicate.

Quantified premise of an exclusive disjunction, and its putative *or*-deletions	Intuitive (system 1) model and its evaluation of *or*-deletions	Deliberative (system 2) models and their evaluation of *or*-deletions	Predicted acceptance (*Yes*) or rejection (*No*) of *or*-deletions from the intuitive model
None of the A's did B or else C. *∴ None of the A's did B/C.*	A A B C Yes: accept	A C A B A C A B B C B C No: reject	Yes: 62
Few of the A's did B or else C. *∴ Few of the A's did B/C.*	A B A C A A A Yes: accept	A B A C A C A B A C A B A C A B A C A B No: reject	Yes: 64
Some of the A's did B or else C. *∴ Some of the A's did B/C.*	A B A C A Yes: accept	A B A C A A C B No: reject	Yes: 68
One of the A's did B or else C. *∴ One of the A's did B/C.*	A B A A No: reject[a]	A C A A No: reject	No: 68
Most of the A's did B or else C. *∴ Most of the A's did B/C.*	A B A B A B A No: reject	A C A C A C A No: reject	No: 76
All of the A's did B or else C. *∴ All of the A's did B/C.*	A B A B A B B No: reject	A C A C A C A C No: reject	No: 84

[a]If the premise is interpreted as not referring to a single particular individual, it has an intuitive model in which one *A* did *B*, and another *A* did *C*, which supports the *or*-deletions. The models above, however, refute this evaluation (see text). Note: The balances of percentages in each cell are for evaluations opposite to the predictions.

do so provided that the quantifier can refer to a proportion of no more than half.

In the case of:

Most of the students chose acting or else dancing

an intuitive process, as we described in the section on quantifiers, constructs a model of the quantifier and the first disjunct, in which most of the students chose acting:

student	chose acting
student	chose acting
student	

The process cannot add the second disjunct—in which most of the students chose dancing—to this model without violating the meaning of the assertion. The quantifier refers to a proportion that is greater than a half—in this case, most of the students—and so a single intuitive model cannot accommodate the alternative disjuncts. The only option is for deliberation to construct a second model to represent the possibility in which most of the students chose dancing. The need for two separate models of mutually exclusive situations shows that the *or*-deletions do not follow from the premise.

In contrast, given a premise such as:

Few of the students chose acting or else dancing

the intuitive construction begins with a model of the quantifier and the first disjunct, such as:

student	chose acting
student	
student	
student	

Next, it adds the interpretation of the quantifier and the second disjunct to the same model:

student	chose acting
student	chose dancing
student	
student	

The quantifier refers to a small enough proportion —in this case, few—for a single intuitive model to accommodate individuals with each of the two mutually exclusive predicates. The model yields acceptance of the following *or*-deletions:

∴ Few of the students chose acting.

∴ Few of the students chose dancing.

Deliberation, however, can yield models that refute both of these conclusions. The premise has an explicit model that refutes the first *or*-deletion above:

student	chose acting
student	chose acting
student	chose acting
student	chose dancing

and an analogous model that refutes the second *or*-deletion. Hence, a quantified exclusive disjunction based on "few" should yield an intuitive acceptance of *or*-deletions that deliberations can refute. The model theory yields:

> Prediction 4. *Or*-deletions from quantified exclusive disjunctions: Participants will tend to accept *or*-deletions more often than not if a quantified disjunction has a single intuitive model of individuals with each of the two mutually exclusive predicates; otherwise, they will tend to reject *or*-deletions more often than not.

Table 4 below shows that quantified exclusive disjunctions based on *none*, *few*, and *some*, which can refer to proportions of less than a half, have intuitive models that should yield acceptances of *or*-deletions, whereas those based on *one*, *most*, and *all*, do not have such models. As Table 4 also shows, deliberative models establish that none of these inferences is valid. A more equivocal case occurs with:

One of the students chose acting or else dancing.

In one interpretation it refers to a particular student, who made the choice, and so its intuitive model represents this same individual as choosing acting, and it calls for alternative model to represent this individual as choosing dancing. The two models lead to the rejection of *or*-deletions. Another, perhaps unlikely, interpretation is that one student chose acting and one student chose dancing, and it can be accommodated within a single model. But, this interpretation cannot overrule the first one, and so individuals should tend to reject *or*-deletions.

No obvious implicatures based on conversational conventions appear to support acceptance of any of the preceding *or*-deletions, and so Gricean theory offers no predictions beyond logic. As we mentioned earlier, post-Gricean theory predicts *or*-deletions for existential quantifiers with plural nouns, e.g. "Some of the students" (Fox, 2007). So, it does not predict that *or*-deletions should follow from "None of the students" or from "Few of the students", because neither is an existential quantifier. No previous experiments had examined *or*-deletions from quantified premises, and so the aim of this experiment was to test these contrasting predictions.

Design and participants

The participants acted as their own controls and evaluated two instances of *or*-deletions from six sorts of quantified exclusive disjunction based on *none*, *few*, *some*, *one*, *most*, and *all*. The design used two complementary assignments of the disjuncts in the predicates to the conclusions, and two different allocations of a set of 12 contents to the inferences. It tested four separate groups of participants of equal numbers with them. The participants' task was to judge whether given the truth of the premise, the conclusion followed from it. They were randomly assigned to one of the four versions of the materials, and each of them received the 12 inferences in a different random order.

The participants were 105 volunteers (80 women and 25 men, mean age 24.5 years $SD = 4.6$) from the same population as before.

Materials and procedure

We created 12 everyday contents of the sort illustrated above (see Table S4 in the supplemental materials). The participants were tested on-line using the Qualtrics site. The key instruction was: "For each problem, given that the premise is true, please judge whether the conclusion follows from it". On each trial, the participants chose their response from two options: *Yes*, or *No*. A typical trial was as follows, where the Portuguese for "or else" is "ou então", and is a strong cue to an exclusive interpretation:

> Some of the musicians played jazz or else rock.
>
> Does it follow that some of the musicians played jazz?
>
> ☐ Yes ☐ No

Results and discussion

Overall, 87 out of the 105 participants made more predicted than unpredicted judgments, 6 made more unpredicted judgments than predicted ones, and the remaining 12 were ties (Binomial test, $p < .00001$). No reliable difference occurred in the percentages of predicted inferences among the four groups with different assignments of material or predicates in the conclusions (71%, 71%, 68%, and 71% predicted inferences, Mann-Whitney tests, z's $< .95$, p's $> .81$, Cliff's $\delta < .15$). We, therefore, amalgamated their results for analysis. Table 4 presents the intuitive and deliberative models of the six sorts of inference, and the percentages of their predicted evaluations. The participants tended to concur in their evaluations over the six sorts of inference (Kendall's co-efficient of concordance, $W = .7$, $p < .001$). Their agreement was low and unreliable for the three inferences that tended to yield *or*-deletions ($W < .1$, $p > .45$), but larger and reliable for the three inferences that tended not to yield them (Kendall's co-efficient of concordance, $W = .4$, $p < .001$). In retrospect, it is clear why this difference should occur, and why the percentages of predicted acceptances tended to be smaller than the percentages of predicted rejections. None of the inferences is valid in the model theory or in logic. But, premises based on *none*, *few*, and *some*, have an intuitive model that yields the inferences and deliberative models that refute them, whereas premises based on *one*, *most*, and *all*, have intuitive models that refute them. Hence, participants are more likely to disagree about the first set of inferences than the second set of inferences. The bias towards treating disjunctions as referring to a single model can be powerful. A reviewer, who remains anonymous here, argued that our results reflected nothing more than logic, and that, for example, the quantified premise based on "few" and an exclusive disjunction in its predicate yields valid *or*-deletions. He erred in a way that the model theory predicts, considering a single intuitive model, and overlooking the counterexamples that deliberation and higher-order logic yield. Logic cannot predict the systematic errors that the participants made, and the results are also contrary to Gricean and to post-Gricean theory (e.g. Fox, 2007). Participants tended to accept *or*-deletions based on "none" and those based on "few" even though neither is an existential quantifier, and to reject those based on "most" even though it is plural and has existential force.

General discussion

Logicians discovered long ago certain disjunctive paradoxes, which we dub *or*-deletions, e.g.:

> Viv may go to Shannon or to Dublin.
>
> ∴ Viv may go to Shannon.

This inference is a paradox because on the one hand, its validity seems obvious, but on the other hand, it is invalid in all normal modal logics, of

which a countable infinity exists. And so a vast literature has developed to try to resolve these paradoxes. Many theories are pragmatic and aim to maintain normal logics in which "or" is truth-functional, i.e. *A or B or both* is true provided that at least one of its clauses is true. As we described in our review, these theories originated in Grice's (1989) account of implicatures arising from the conventions of conversation. Subsequent researchers devised ingenious extensions and variants of his resulting "conversational implicatures" to try to salvage modal logics and their truth-functional semantics of sentential connectives. They formalised his ideas (e.g. Gazdar, 1979), used them to explain *or*-deletions (e.g. Kratzer & Shimoyama, 2002), and converted them into radical post-Gricean theories (e.g. Fox, 2007). In contrast, Zimmermann (2000) and Geurts (2005) abandoned truth-functional semantics in order to cope with *or*-deletions. The theory of mental models—the model theory—aims to explain human reasoning in general (Johnson-Laird, 2006), and its most recent version follows these precursors and abandons truth-functional semantics. It treats disjunctions as referring to an exhaustive conjunction of possibilities that each holds in default of knowledge to the contrary (Byrne & Johnson-Laird, 2019; Hinterecker et al., 2016; Khemlani et al., 2018). These possibilities are based on finite primordial alternatives from which one of them is bound to occur (Johnson-Laird & Ragni, 2019), but these core possibilities are defaults, and so particular interpretations, such as deontic permissions or epistemic possibilities, can replace them. The resulting modal disjunctions are not primordial, and so they can yield *or*-deletions, such as the example heading this discussion.

A modal disjunction of the sort above:

Viv may go to Shannon or to Dublin

yields intuitive models of a conjunction of Viv's possible destinations, which each hold in default of knowledge to the contrary:

Shannon

Dublin

It the disjunction is interpreted as inclusive, it adds a model of their joint possibility:

Shannon Dublin

The disjunction also presumes that Viv may not go to these destinations. But, the three models above yield the *or*-deletions:

∴ Viv may go to Shannon.

∴ Viv may go to Dublin.

∴ Viv may go to Shannon and to Dublin.

These inferences are valid in the model theory whether the disjunction is interpreted as referring to deontic permisions or to epistemic possibilities, or both.

The model theory leads to four main predictions about *or*-deletions, and our experiments corroborated them:

(1) *Free choice permissions*. Individuals accept *or*-deletions from free choice permissions, whether a disjunction has a narrow scope within a single permission:

You are permitted do A or B;
∴ You can do A

or a wide scope over two permissions:

You are permitted do A or you are permitted to do B;
∴ You can do A

where *A* and *B* stand for sensible everyday actions (Experiment 1).

(2) *Epistemic* or-*deletions and* or-*introductions*. Individuals accept epistemic *or*-deletions:

It is possible that A did B or C
∴ A may have done B

from all sorts of disjunction, narrow or wide scope, and inclusive or exclusive, and they accept them much more often than they accept their converese *or*-introductions, which are valid in normal modal logics:

It is possible that A did B

∴ A may have done B or C

where *A* denotes an actor, and *B* and *C* are sensible actions (Experiment 2).

(3) *Truth of conditionals asserting valid* or-*deletions*. Individuals evaluate as true those conditionals embodying *or*-deletions that are valid in the model theory, such as those of the sort:

If it is possible that A or that B then it is possible that A (Experiment 3).

(4) Or-*deletions from quantified exclusive disjunctions*. Participants tend to accept *or*-deletions from a disjunction that has a single intuitive model of individuals and their mutually exclusive predicates, e.g. those based on *none*, *few*, and *some*, such as:

Few of the A's are B or else C..∴ Few of the A's are B.

They tend to reject *or*-deletions from quantified exclusive disjunctions that cannot have such models, e.g. those based on *one*, *most*, and *all*, such as:

Most of the A's are B or else C.

∴ Most of the A's are B.

where *A* refers to a set of agents and *B* and *C* refer to actions (Experiment 4).

A fifth prediction was corroborated in a recent study:

(5) *Metadisjunctions can yield* or-*deletions*. Individuals tend to accept *or*-deletions from metadisjunctions (see our earlier account of them), such as:

Witness 1 stated: It is possible that the defendant was trespassing in the victim's garden.

Witness 2 stated: It is possible that the defendant was on business in the victim's garden.

One witness was telling the truth, and one witness was lying.

Is it possible that the defendant was trespassing in the victim's garden?

In the study all 60 participants answered, "yes", thereby making an *or*-deletion, and as the model theory also predicts they answered "no" to other problems (Sklarek et al., 2021). These sorts of *or*-deletion cannot depend on Gricean implicatures, which apply only to single assertions as a whole (Cohen, 1971). Likewise, the post-Gricean theory cannot yield them, because its exhaustion operator applies only to a single sentence, not to a disjunction formed out of three sentences. Bar-Lev (p.c., 1/17/21) conceded that the theory is silent about metadisjunctions, but suggested that they should not yield *or*-deletions from permissions. We carried out a recent unpublished study that shows that such *or*-deletions do occur.

Table 5 summarises these five corroborated predictions and their consequences for the principal accounts of *or*-deletions. We have not tried to distinguish actual predictions of a theory from its potential predictions. The results show that normal modal logics are remote from human reasoning, and that Gricean and post-Gricean theories fail to account for some of them too.

The historical accident that *or*-deletions were first discovered for deontic possibilities and then for epistemic possibilities may have led to too great a focus on them. Other sorts of simpler assertion can elicit *or*-deletions, e.g.:

Table 5. Five of the model theory's predictions corroborated in experiments, and their consequences for theories of *or*-deletion: ◇ symbolises a possibility, + indicates that a theory predicts or has the potential to predict a result, – indicates it cannot predict at least one of the relevant results, and A, B, and C denote sensible everyday contents.

Theories	Experiment 1 Deontic free choices occur for narrow- and wide-scope "or"	Experiment 2 Epistemic *or*-deletions occur more often than *or*-introductions	Experiment 3 Conditionals containing *or*-deletions judged to be true	Experiment 4 Quantified *or*-deletions tend to occur from single intuitive models	Sklarek et al. (2021) Epistemic metadisjunctions yield *or*-deletions
	◇(A or B). ∴◇A. ◇A or ◇B. ∴◇A.	◇(A or B). ∴◇A. vs: ◇A ∴◇(A or B).	If ◇(A or B) then ◇A.	Few A are B or C. ∴ Few A are B. vs. All A are B or C. ∴ All A are B.	One is true & one is false: ◇A. ◇B. ∴ ◇A.
Normal modal logics	–	–	–	–	–
Gricean implicatures	+	–	–	–	–
Post-Gricean implicatures	–	–	+	–	–
Mental model theory	+	+	+	+	+

Note: The acceptances of *or*-deletions are illustrated with a single example of a conclusion.

Jeff is richer than Elon or Bill.

∴ Jeff is richer than Elon.

and perhaps the simplest of all:

Jancis likes red or white wine.

∴ Jancis likes red wine.

Other sorts of connective may also create analogues of them. These are topics for future research.

Conclusions

The paradoxes of *or*-deletion seemed at first to be a minor anomaly: *you can read the literature or you can skip it*; so, *you can skip it*. In fact, they occur in many domains. Our studies show that a semantic solution to these anomalies is more plausible than pragmatic ones. The results do not undermine the fact that conventions governing discourse lead to interpretations beyond the literal sense of sentences. But, they do suggest that despite its long tradition from Stoic logicians through Boolean algebra, truth-functional semantics is only a surface account of the meanings of disjunctions and other connectives. And the logical apparatus of possible worlds is an elegant colossus—too big to be at home in the brain. The corroboration of five principal predictions supports another explanation. Small sets of finite alternatives—of which one is bound to occur—are primordial. They are represented in mental models, and underlie possibilities and probabilities. Disjunctions refer to conjunctions of such possibilities that each hold in default of knowledge to the contrary. A primordial disjunction does not allow *or*-deletions. But, a simple way to convey that one of a set of alternatives is not bound to happen is to assert that they are only possibilities; another way is to use a quantified exclusive disjunction that has a single intuitive model accommodating both mutually exclusive predicates. They then yield *or*-deletions. Everyday inferences diverge from orthodox logics in many other ways. Yet, as logicians have long understood, their discipline is not a theory of human reasoning.

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No potential conflict of interest was reported by the author(s).

Data availability and deposition

The supplemental materials for the four experiments and their datasets are available at: https://osf.io/p7ucf/?view_only=e0d349a6770e4a56a3ce830296d2af8e.

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